The Theorems of Green, Gauss (Divergence), and Stokes

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Green’s Theorem:

Let $R$ be a simply connected region with a piecewise smooth boundary $C$, oriented counterclockwise. If $M$ and $N$ have continuous partial derivatives in an open region containing $R$, then

$$\oint_{C} M \, dx + N \, dy = \iint_{R} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA$$
Green's Theorem (alt. form):

We have:

\[ \oint_C M \, dx + N \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA \]

Now, replace the unit tangent vector \( \mathbf{T} \) with the outward unit normal \( \mathbf{N} = \langle y'(s), -x'(s) \rangle \) (see p. 1099). We then have:

\[ \oint_C (-N) \, dx + M \, dy = \iint_R \left( \frac{\partial M}{\partial x} - \frac{\partial (-N)}{\partial y} \right) \, dA = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \, dA \]

or

\[ \oint_C \mathbf{F} \cdot \mathbf{N} \, ds = \iint_R \text{div} \, \mathbf{F} \, dA \]
Now, move from two dimensions to three. Thus, the line integral around the boundary \( C \) of a region \( R \) in the plane becomes a **surface integral** over the surface \( S \) of a solid region \( Q \) in space, and the double integral in the plane becomes a **triple integral** in space. I.e.;

\[
\oint_{C} \mathbf{F} \cdot \mathbf{N} \, ds = \iint_{R} \text{div} \mathbf{F} \, dA
\]

becomes **Gauss’ (Divergence) Theorem:**

\[
\iiint_{S} \mathbf{F} \cdot \mathbf{N} \, dS = \iiint_{Q} \text{div} \mathbf{F} \, dV
\]
Again, Green’s Theorem:

Let \( R \) be a simply connected region with a piecewise smooth boundary \( C \), oriented counterclockwise. If \( M \) and \( N \) have continuous partial derivatives in an open region containing \( R \), then

\[
\oint_C M \, dx + N \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA
\]
We again move from two dimensions to three, but this time in a slightly different but, perhaps, a more “natural” manner. We simply let the closed curve $C$ lift off of the plane to become the boundary of a patch of surface $S$ in space. But first we rewrite Green’s Theorem from

$$
\oint_C M \, dx + N \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA
$$

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Now, moving off the $x$-$y$ plane, where

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \text{curl} \, \langle M, N, 0 \rangle \cdot \langle 0, 0, 1 \rangle$$

to 3-space where

$$\text{curl} \, \mathbf{F} = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle,$$

we get **Stokes’ Theorem:**

$$\oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{S} \text{curl} \, \mathbf{F} \cdot \mathbf{N} \, ds$$