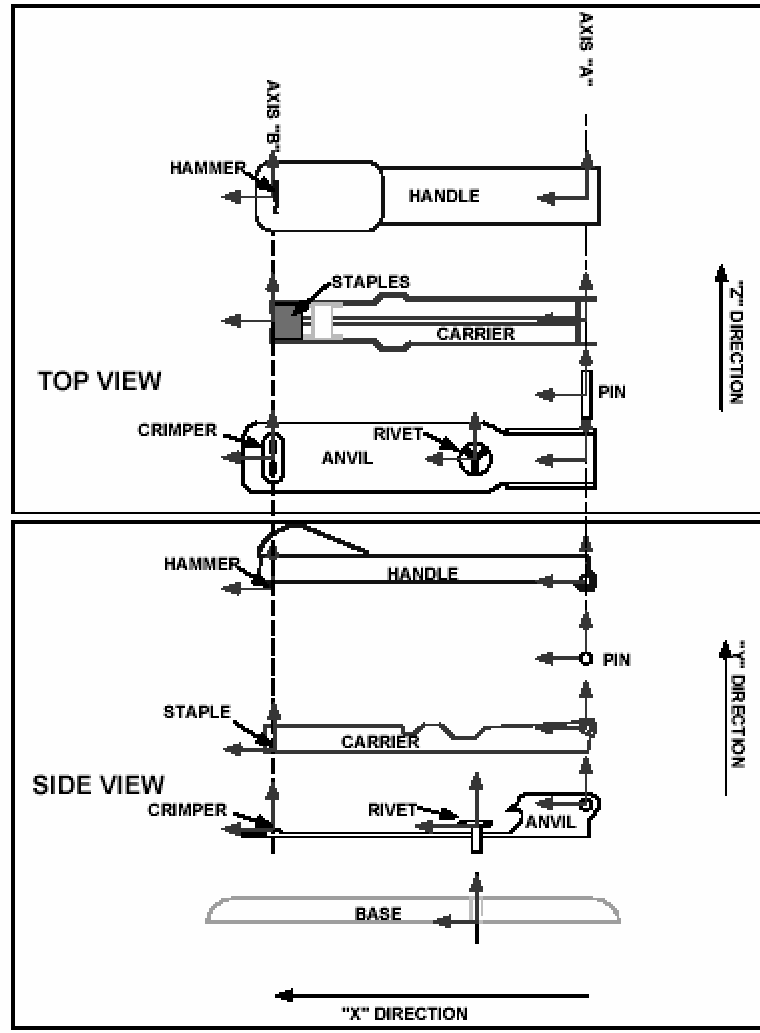
Mathematical Modeling of Assembly

- Coordinate frames
  - each part has a base coordinate frame
- Relationships between parts are expressed as 4x4 matrix transforms



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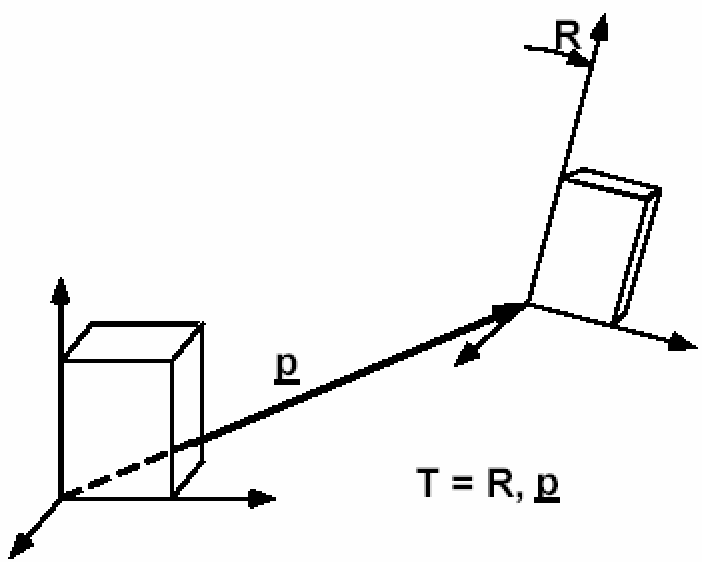
Matrix Math

- 4x4 matrices relate adjacent frames
- Matrix contains rotational part and translational part
- Translation occurs first, so rotation does not change position of new frame

Basic Translation and Rotation



$$T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} R & p \\ 0^T & 1 \end{bmatrix}$$

Watch Transform Ordering!

$$T = \begin{bmatrix} R & p \\ 0^T & 1 \end{bmatrix}$$

$$= \begin{bmatrix} I & p \\ 0^T & 1 \end{bmatrix} * \begin{bmatrix} R & 0 \\ 0^T & 1 \end{bmatrix}$$

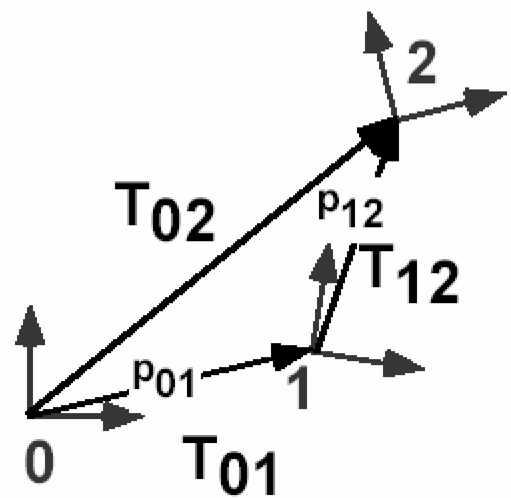
$$\neq \begin{bmatrix} R & 0 \\ 0^T & 1 \end{bmatrix} * \begin{bmatrix} I & p \\ 0^T & 1 \end{bmatrix} \quad !!!$$

Composite Transforms

$$T_{02} = T_{01}\, T_{12}$$

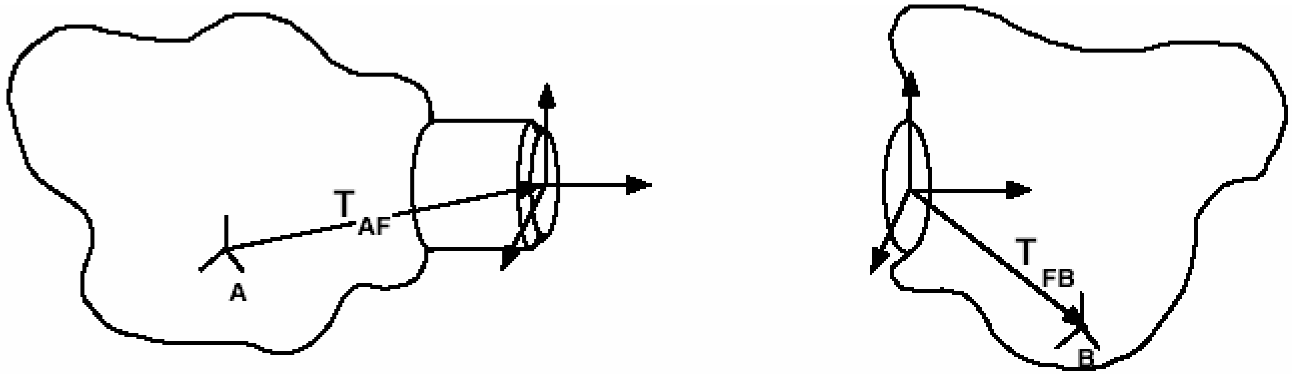
$$T_{02} = \begin{bmatrix} R_{01} & p_{01} \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} R_{12} & p_{12} \\ 0^T & 1 \end{bmatrix} =$$

$$\begin{bmatrix} R_{01}R_{12} & R_{01}\, p_{12} + p_{01} \\ 0^T & 1 \end{bmatrix}$$

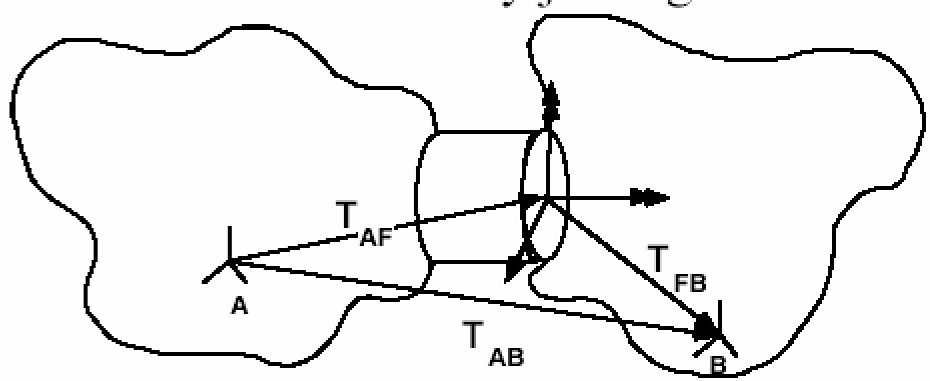


T_{01} locates frame 1 in frame 0 coordinates
T_{12} locates frame 2 in frame 1 coordinates
T_{02} locates frame 2 in frame 0 coordinates

Nominal Mating of Parts



Parts A and B are mated by joining two features



The nominal location of part B can be calculated from the nominal location of part A using 4x4 transform math

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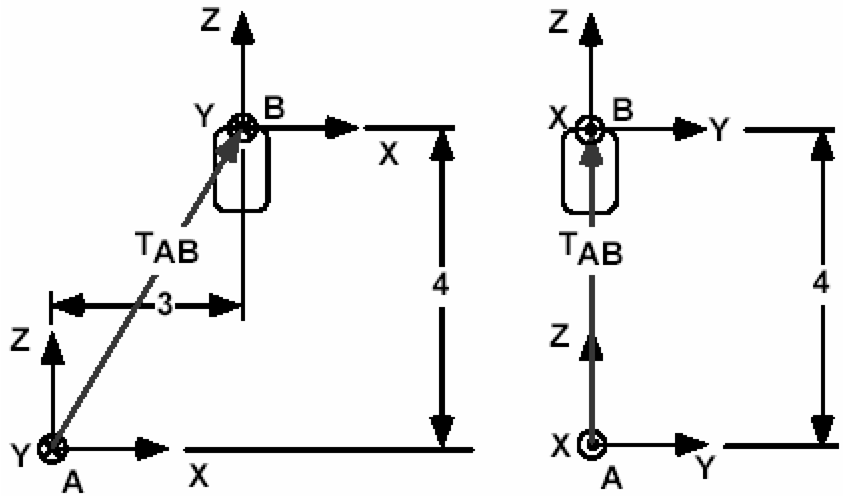
Example - Pin & Hole Mating (pin translated)



$$>> T_{AB} = trans(3,0,4)$$

$$T_{AB} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Coordinate Frames

MATLAB(TM) Code

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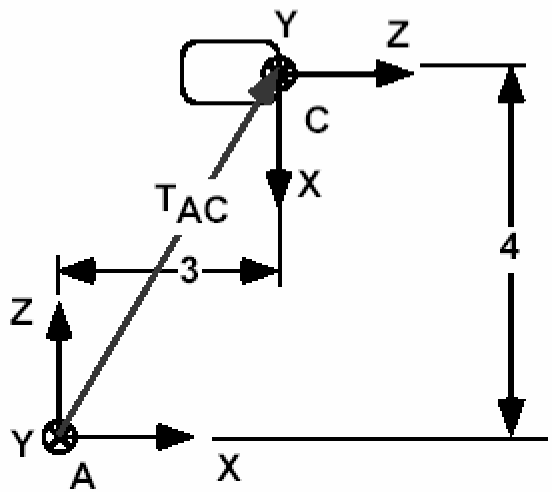
Example - Pin & Hole Mating (pin rotated)

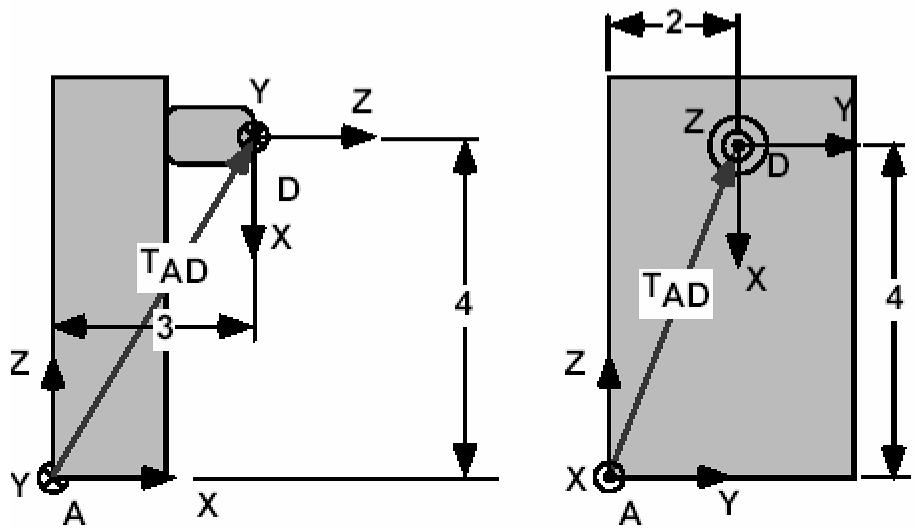


$$>> T_{AC} = T_{AB}\, roty\,(dtr\,(90))$$

$$T_{AC} = \begin{bmatrix} 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
function degtorad = dtr(theta)
% Converts degrees to radians
degtorad=theta*pi/180;
```

Example - Pin & Hole Mating (feature on first part)



$$>> T_{AD} = trans(3,2,4)\,roty\,(dtr\,(90))$$

$$T_{AD} = \begin{bmatrix} 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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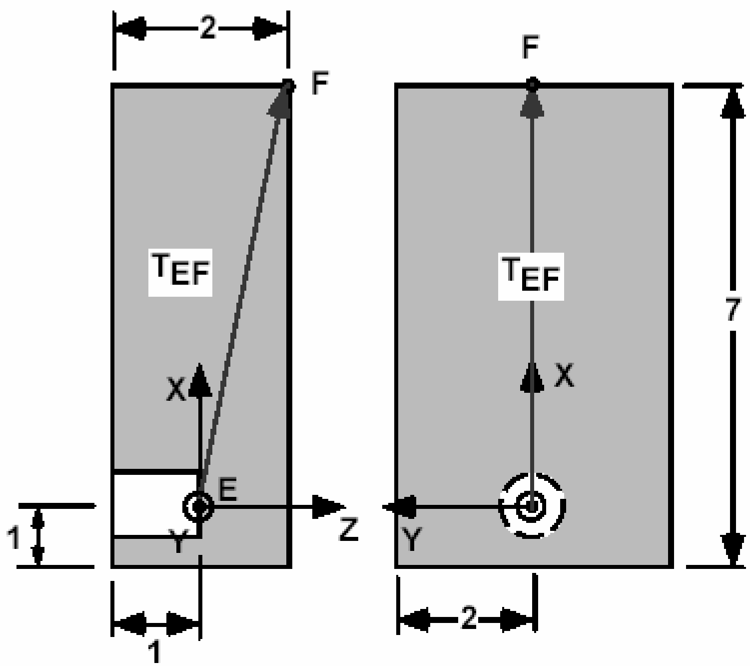
Example - Pin & Hole Mating (feature on second part)



$>> T_{EF} = trans\,(6,0,1)$

$$T_{EF} = \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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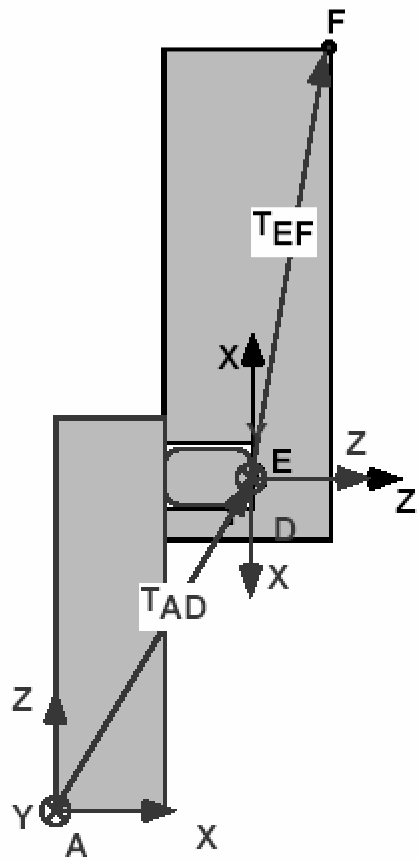
Example - Pin & Hole Mating (Assembling two parts)



$$>> T_{DE} = rotz\ (dtr(180))$$

$$T_{DE} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$>> T_{AF} = T_{AD} T_{DE} T_{EF}$$

$$T_{AF} = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4x4_examples copy

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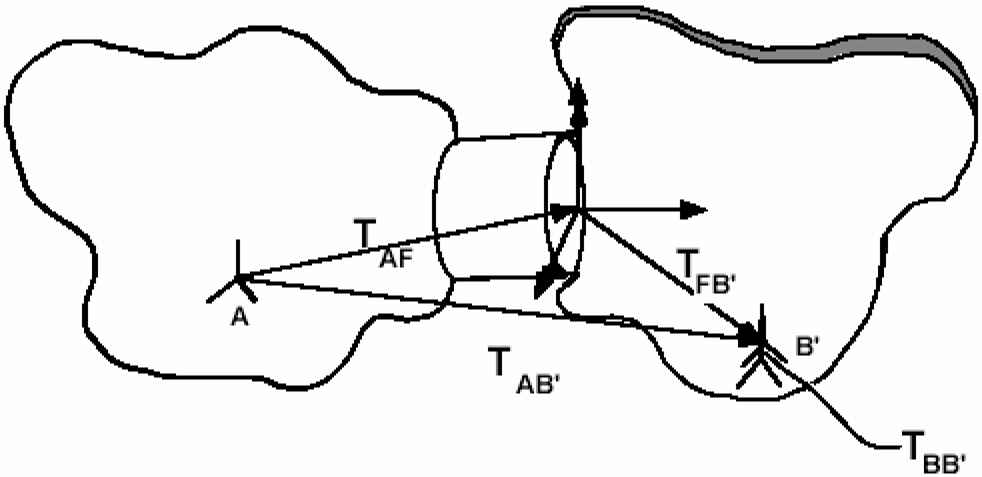
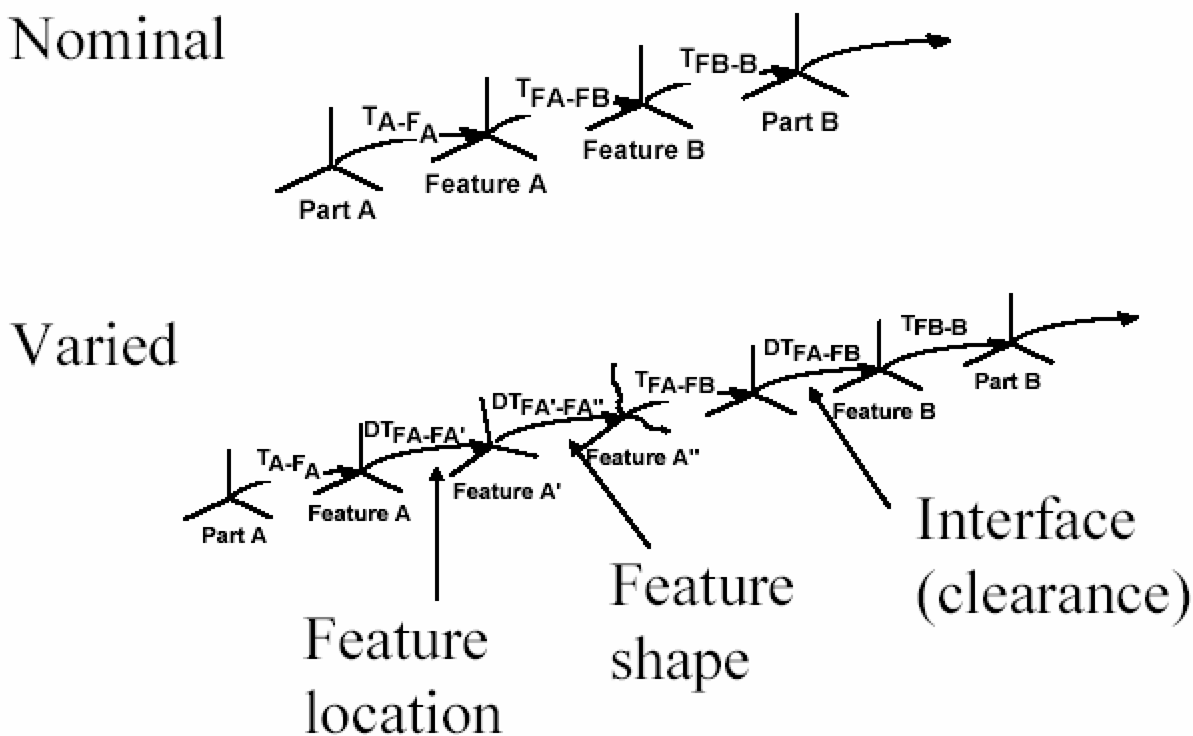
Part Location Variation



- Varied location of Part B calculated from nominal location of Part A
- Uses same math as nominal model!

Chaining together parts



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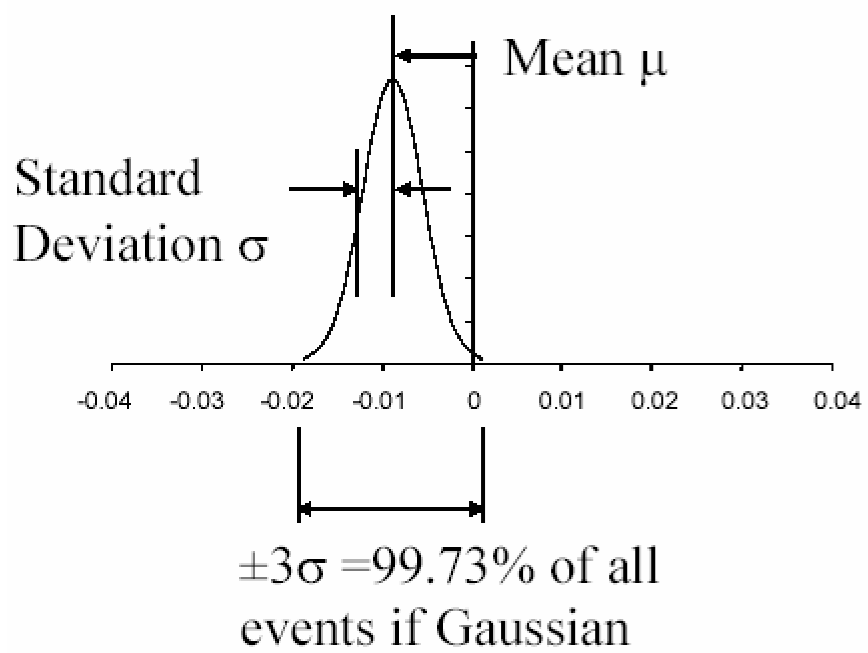
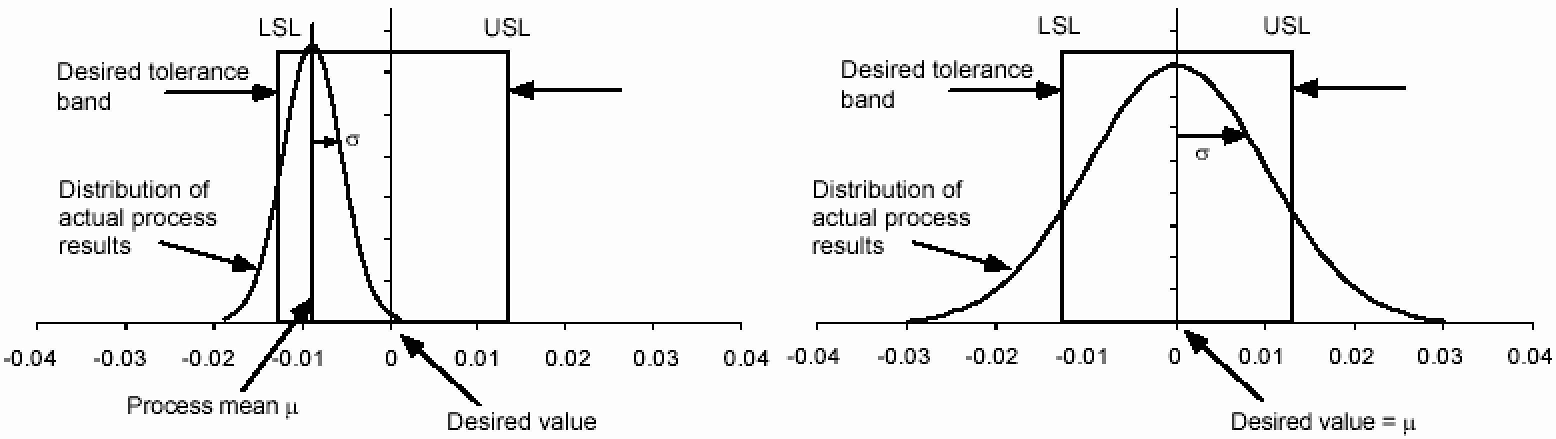
Variation Analysis

- Error types:
  - Change in process average (mean shift)
  - Process variation around average (variance)



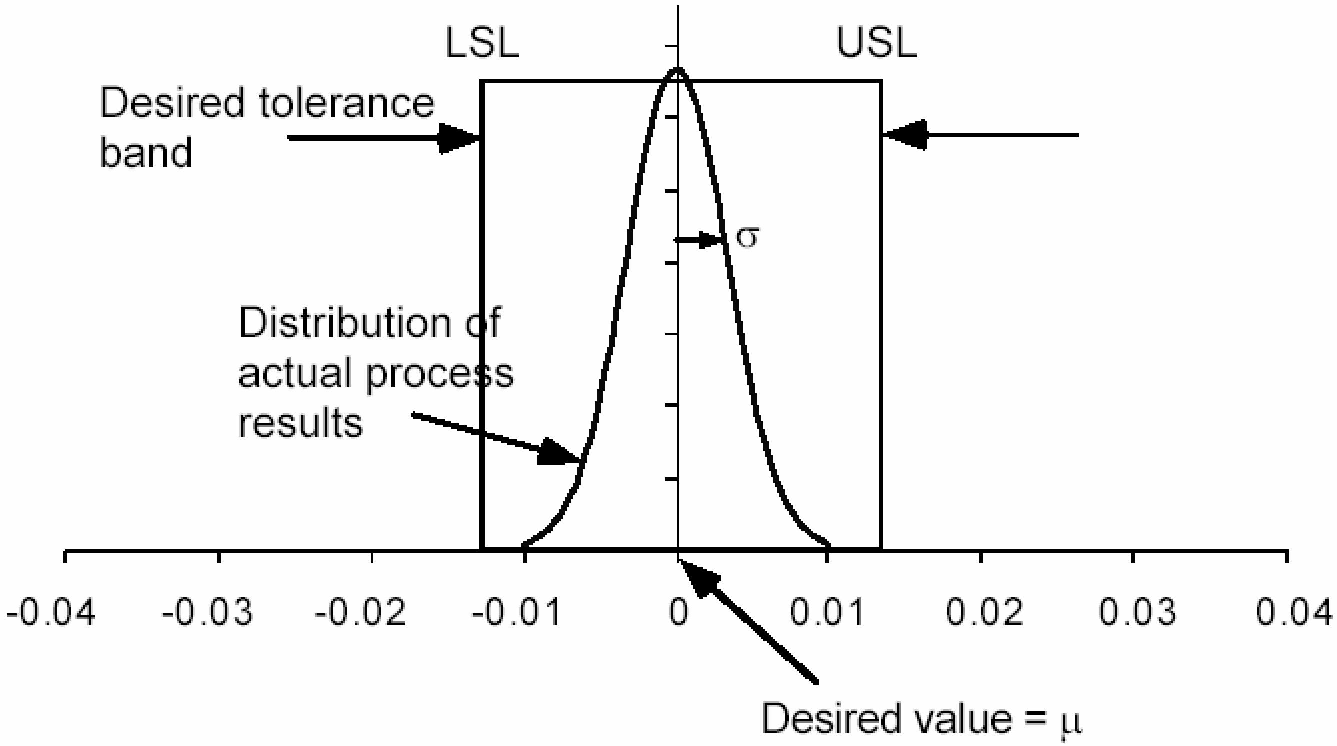
Precision vs. Accuracy



Consistent but consistently wrong

Right on the average

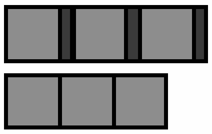
Desired Distribution



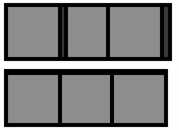
- LSL = lower specification limit
- USL = upper specification limit

Error Accumulation

- Worst case tolerancing:



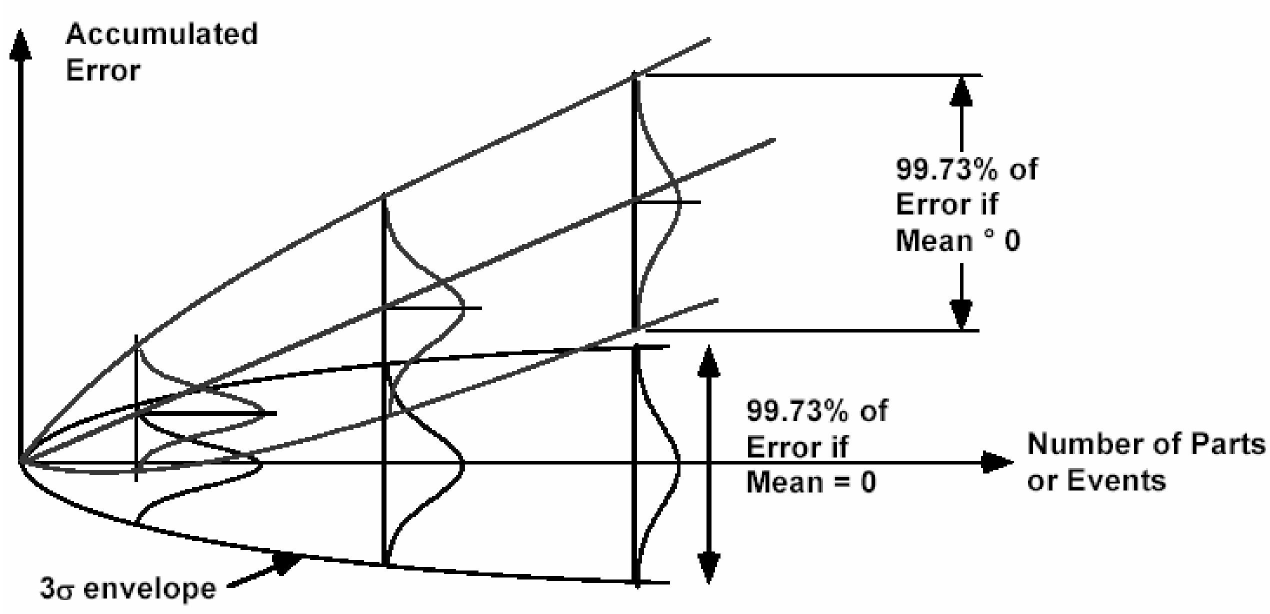
  - assume all errors at extremes
  - errors accumulate linearly w/ # of parts
  - deterministic, not statistical
- Statistical tolerancing
  - assume errors distributed randomly between limits
  - errors accumulate as sqrt of # of parts



    - if mean is equal to nominal dimension!

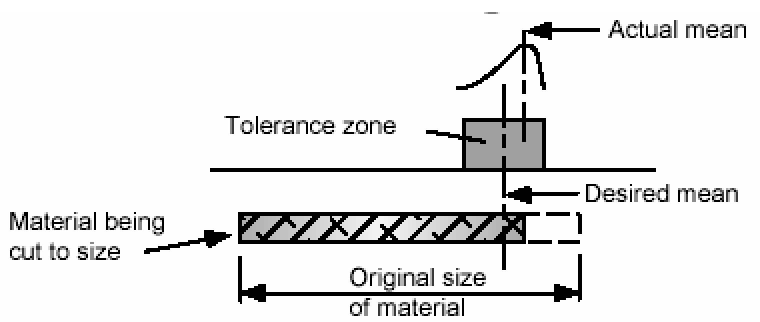
Error Accumulation

- Sums of zero-mean errors accumulate as sqrt(N), because + and - errors cancel
- Sums of non-zero-mean errors accumulate as N, because there are no cancellations

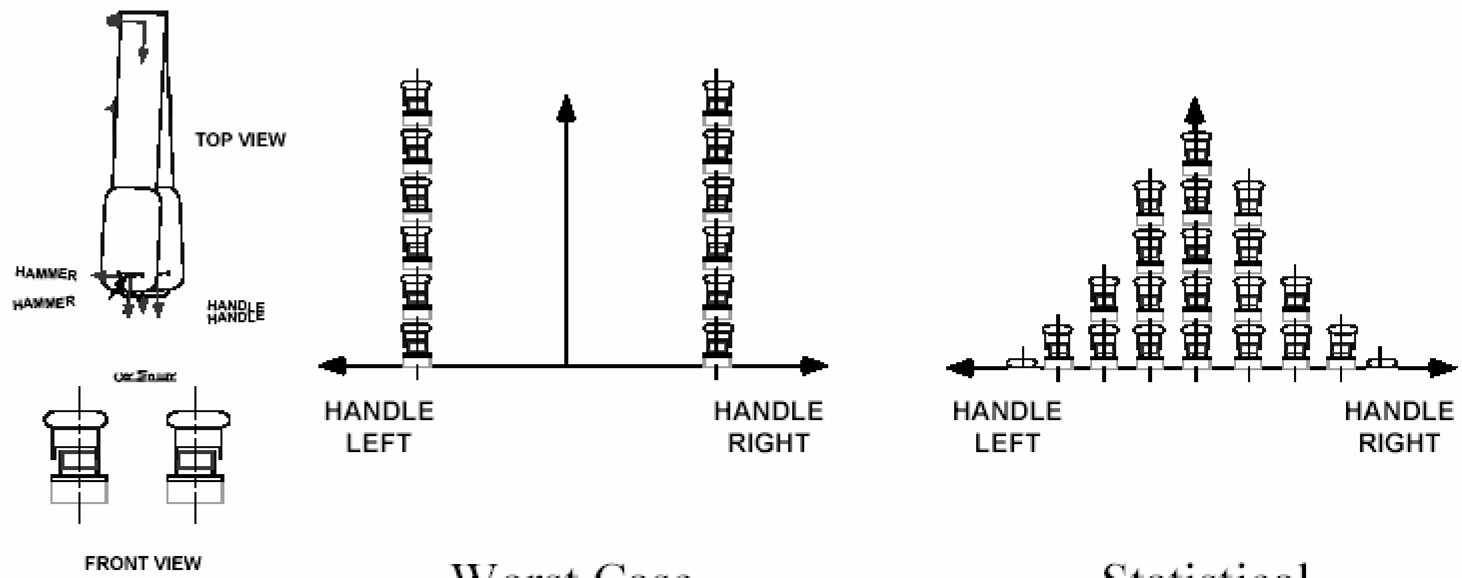


How do Non-zero-mean errors occur?

- Example:
  - operator stops material removal as soon as part enters tolerance zone



Example - Stapler Variations

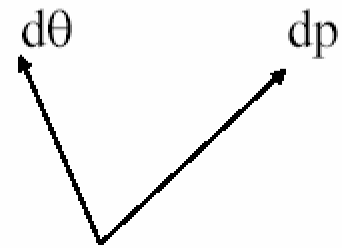


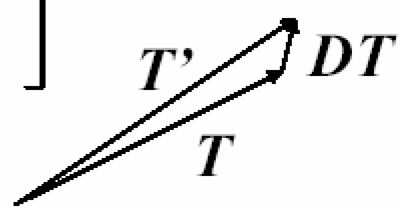
Worst Case

Statistical

(small) Error Transform

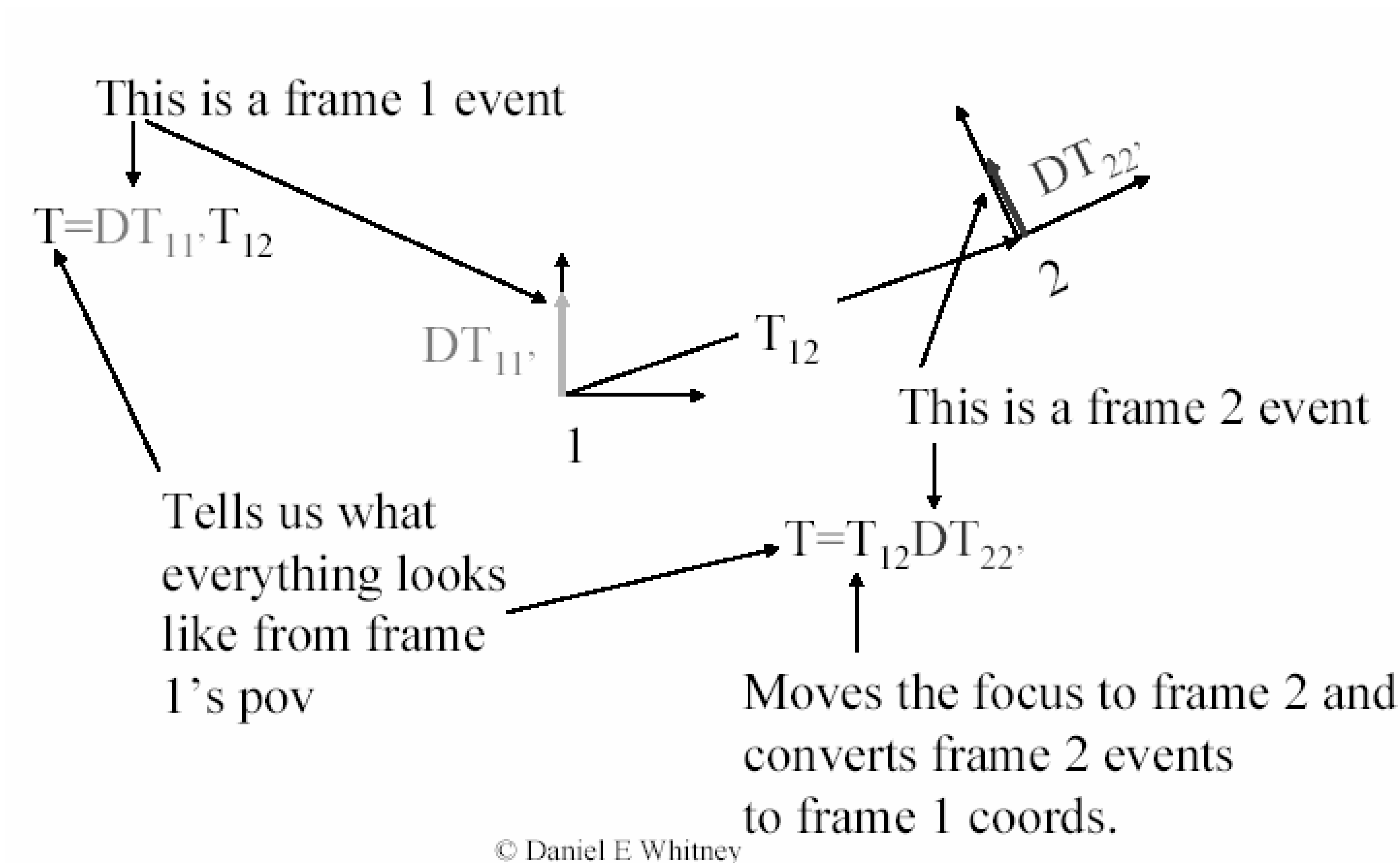
$$DT = \begin{bmatrix} 1 & -\delta\theta_z & \delta\theta_y & dx \\ \delta\theta_z & 1 & -\delta\theta_x & dy \\ -\delta\theta_y & \delta\theta_x & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



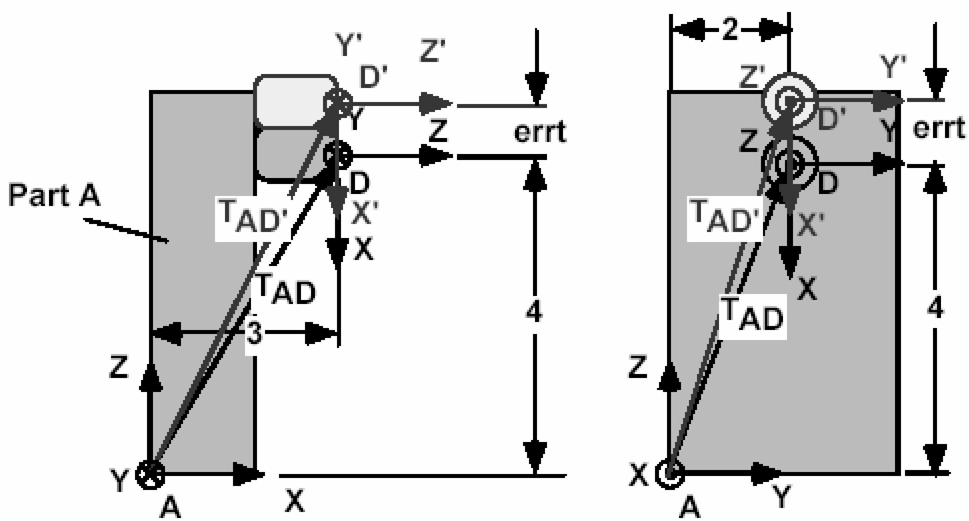


$$T' = T * DT$$

Using Error Transform



Using Error Transform



$>> T_{AD} = trans(3,2,4) * roty(dtr(90))$

% *dtr* converts degrees to radians

$$= \begin{bmatrix} 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

% First method

$>> DZ = errt$

$>> DT_{AD1} = trans(0,0,DZ)$

$T_{AD'1} = DT_{AD1} T_{AD}$

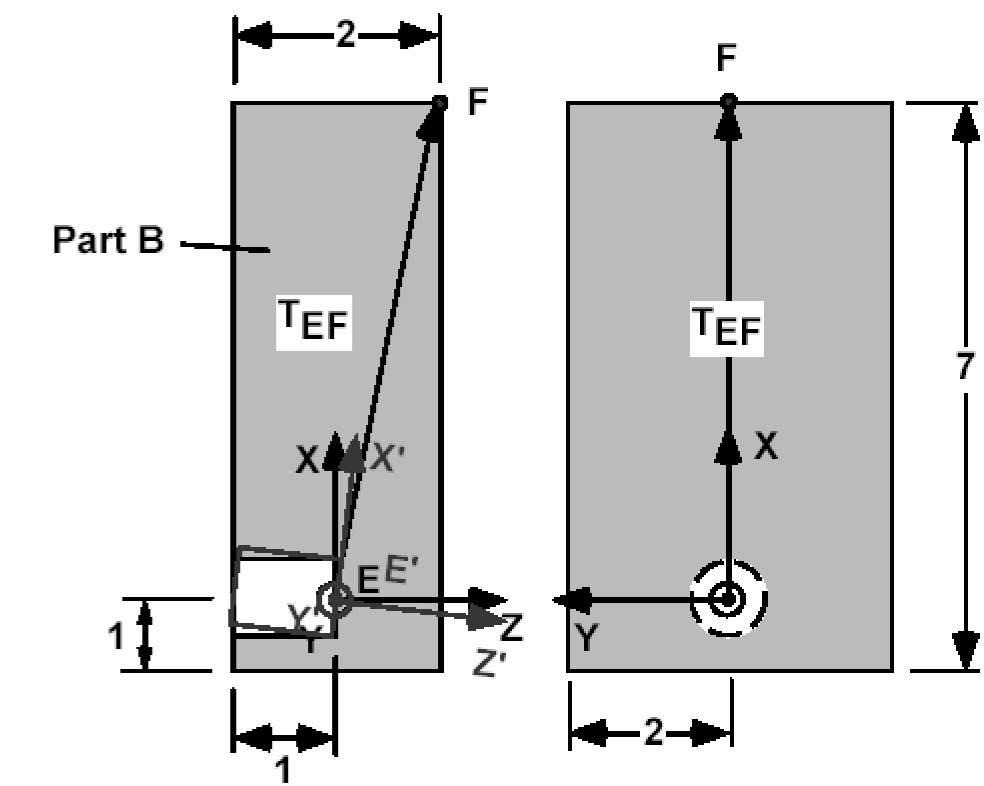
% Second method

$>> DX = errt$

$>> DT_{AD2} = trans(-DX,0,0)$

$>> T_{AD'2} = T_{AD} DT_{AD2}$

Using Error Transform - Part 2



$$>> T_{EF} = trans\ (6,0,1)$$

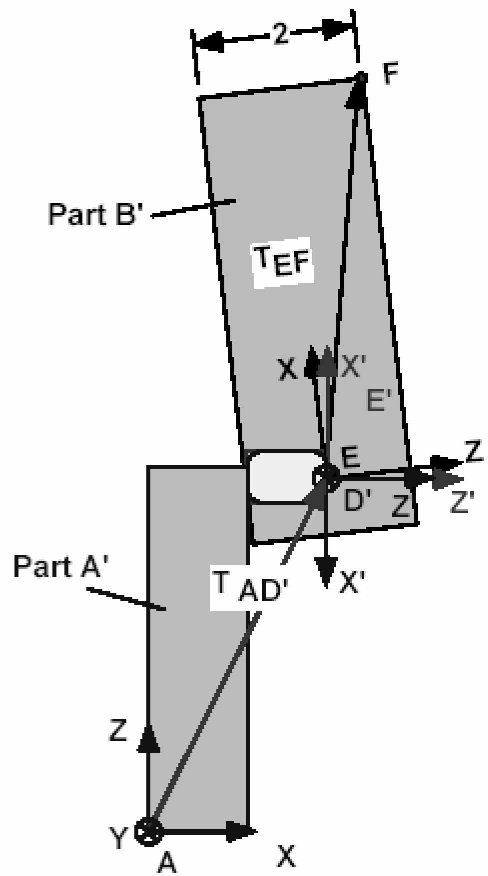
$$= \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$>> T_{EE'} = roty\ (dtr\ (-erra))$$

$$>> T_{E'E} = roty\ (dtr\ (erra))$$

$$>> T_{E'F} = T_{E'E} T_{EF}$$

Using Error Transform - Part 2



$>> T_{DE} = rotz(dtr(180))$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$>> T_{AF} = T_{AD} T_{DE} T_{EF}$

$$= \begin{bmatrix} 0 & 0 & 1 & 4 \\ 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$>> T'_{AF} = T_{AD} . T_{D'E'} . T_{E'F}$

$\% T_{D'E'} = T_{DE}$

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