Stiffness matrix for 2D tapered beams
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1 Introduction

This article presents information necessary for the construction of the stiffness matrix of a typical two-dimensional beam. Information to extend the approach to tapered beams is also presented. The stiffness matrix created is then ready for use in a standard 2D linear elastic frame analysis program. In order to understand the method used to obtain the stiffness matrix the following items are discussed:

1. The concept of complete degrees of freedom and natural degrees of freedom.
2. The relationship between complete degrees of freedom and natural degrees of freedom.
3. Flexibility matrix of the natural degrees of freedom.
4. The equivalence of virtual work in the complete and natural degree of freedom systems.
5. Extension to the case of a tapered beam of I-shaped cross-section.

Following the above information a specific example is studied with numerical results given.

2 Complete and natural degrees of freedom

A typical 2D beam element has 6 degrees of freedom with chosen positive directions as shown in Figure 1a. This is a 2D beam, free in space, with no supports. Each end of the beam has one rotational and two translational degrees of freedom. In the unrestricted state, these are the beam's complete degrees of freedom.

Alternatively, suppose the unrestricted beam is provided with supports. The number of supports provided are such that the resulting beam and support system is statically determinate. The unrestrained degrees of freedom, of this new beam and support system, are called the natural degrees of freedom. A simple way of providing supports that meet the necessary requirements is illustrated in Figure 1b with chosen positive directions for the degrees of freedom shown. A different choice of supports and natural degrees of freedom is possible, however, the ones indicated in the figure prove to be convenient for the ensuing developments.

3 Relation between complete and natural degrees of freedom

It is possible to construct a transformation matrix that provides a relationship between the complete degrees of freedom and the natural degrees of freedom. The complete degrees of

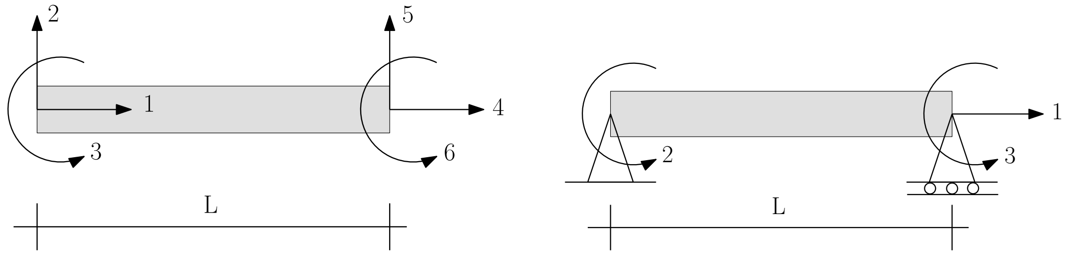

Figure 1: 2D Beam Element. (a) complete degrees of freedom; and (b) natural degrees of freedom.

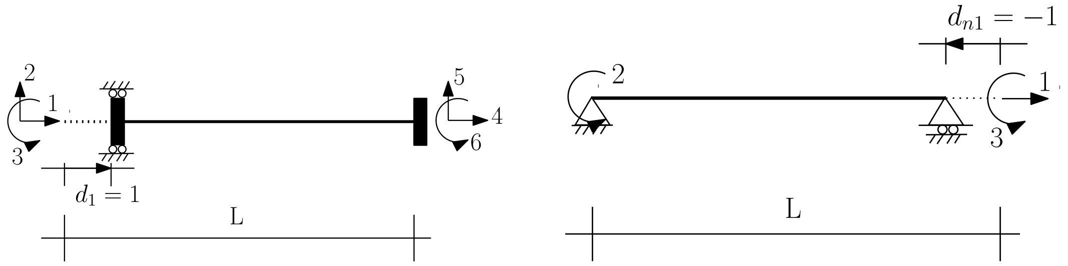

Figure 2: Unit displacement at complete degree of freedom 1. (a) complete degrees of freedom; and (b) natural degrees of freedom.

freedom, $\mathbf{d}$, are represented as a 6 by 1 column vector. The natural degrees of freedom, $\mathbf{d}_n$, are represented as a 3 by 1 column vector. An n is used to indicated those variables associated with the natural degrees of freedom. Using this notation the relationship between the natural degrees of freedom and the complete degrees of freedom is:

$$\mathbf{d}_n = \boldsymbol{\Gamma}\mathbf{d}, \tag{1}$$

where $\boldsymbol{\Gamma}$ is the 3 by 6 transformation matrix. In order to construct the transformation matrix each complete degree of freedom is considered one at a time. The 6 columns of the transformation matrix correspond to the 6 complete degrees of freedom.

First, consider complete degree of freedom 1. Suppose complete degree of freedom 1 is given a unit displacement while all other complete degrees of freedom are held at zero. In order to do this the beam element must deform in compression as shown in Figure 2a. Correspondingly, the question is, what displacements are necessary in the natural degree of freedom system to cause the same effect, that is, in this case, axial compression? The answer is that the natural degree of freedom 1 must be displaced in the negative sense a unit distance as seen in Figure 2b. This is the only way that a unit compressive displacement can be caused in the natural degree of freedom beam. In addition, for this case the natural degrees of freedom 2 and 3 are zero. Hence the first column of the transformation matrix is $[-1 \quad 0 \quad 0]^T$.

Second, consider complete degree of freedom 2. Suppose complete degree of freedom 2 is given a unit displacement while all other complete degrees of freedom are held at zero. In order to do this the beam element must deform in flexure as shown in Figure 3a. The

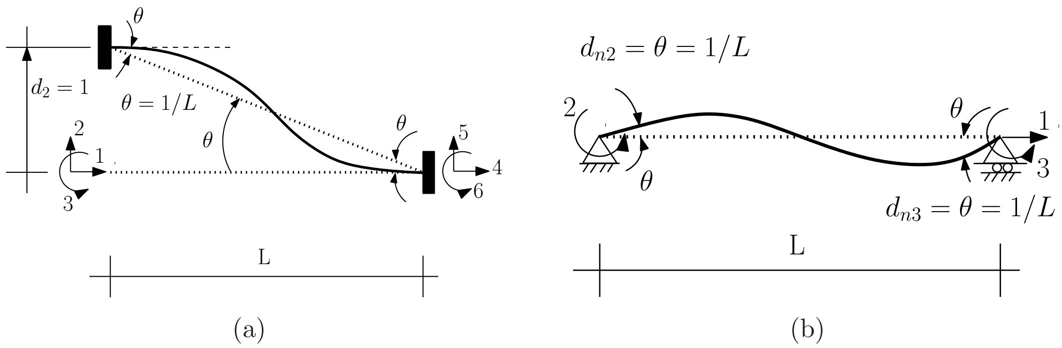

Figure 3: Unit displacement at complete degree of freedom 2. (a) complete degrees of freedom; and (b) natural degrees of freedom.

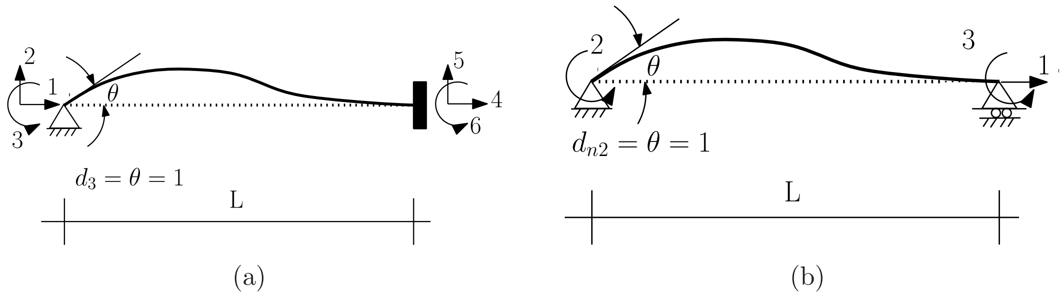

Figure 4: Unit displacement at complete degree of freedom 3. (a) complete degrees of freedom; and (b) natural degrees of freedom.

question is, what displacements are necessary in the natural degree of freedom system to cause the same effect? In this case, it is necessary to have each end rotated an angle of $1/L$ as shown. These rotations are in the positive sense measured from a straight line connecting the start node and end node of the beam element. These necessary rotations cause the beam in the natural degree of freedom system to undergo flexure identical to that taking place in the complete degree of freedom system. As a result the second column of the transformation matrix is $[0 \quad 1/L \quad 1/L]^T$.

Third, consider complete degree of freedom 3. Suppose complete degree of freedom 3 is given a unit rotation while all other complete degrees of freedom are held at zero. In order to do this the beam element must deform in flexure as shown in Figure 4a. The question for this case is, what displacements are necessary in the natural degree of freedom system to cause the same effect? In this case, it is necessary to have the left end rotate through a unit angular displacement and the right end is held to a zero angular rotation as shown. The rotation is in the positive sense measured from a straight line connecting the start node and end node of the beam element. These requirements cause the beam in the natural degree of freedom system to undergo flexure identical to that taking place in the complete degree of freedom system. As a result the second column of the transformation matrix is $[0 \quad 1 \quad 0]^T$.

The columns of the transformation matrix associated with complete degrees of freedom

4, 5, 6 are found in a similar way. The resulting transformation matrix, $\mathbf{\Gamma}$, is:

$$\mathbf{\Gamma} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1/L & 1 & 0 & -1/L & 0 \\ 0 & 1/L & 0 & 0 & -1/L & 1 \end{bmatrix}. \tag{2}$$

The given results for, $\mathbf{\Gamma}$, provide a relationship between the complete and natural systems as indicated in equation (1). This relationship is very useful and is used in some of the following sections. Another interpretation is as follows:

$$\mathbf{d}_{nj} = \mathbf{\Gamma}_{ji}\mathbf{d}_i, \tag{3}$$

so that $\mathbf{\Gamma}_{ji} = \mathbf{d}_{nj}$ when $\mathbf{d}_i = 1$ and $\mathbf{d}_m = 0$, for all $m \neq i$. The reader is encouraged to verify the construction of $\mathbf{\Gamma}$. Furthermore, the reader should recognize that the deformations that occur in the complete and natural systems are identical for all cases indicated above with respect to a straight line drawn from the start node to the end node of the beam. Hence, no matter what the orientation of the beam in 2D space the relationship between complete and natural degrees of freedom are valid. Last, it is easy to verify that the results for $\mathbf{\Gamma}$ are the same for straight, tapered and curved beam elements.

4 Natural degrees of freedom flexibility matrix

Consider the natural degree of freedom system shown in Figure 1b. To find the flexibility matrix for this system the method of virtual work is used. Each term in the flexibility matrix is found by use of the following definition: f_{ij} = displacement at degree of freedom i when a unit real force is placed at degree of freedom j and all other degrees of freedom are held at zero. This is the definition commonly used in standard structural analysis text books. The flexibility matrix for the natural system is

$$\mathbf{D}_n = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}. \tag{4}$$

Since the flexibility matrix is symmetric (by Maxwell's Theorem of Reciprocal Displacements) and since $f_{12} = f_{21} = 0$ and $f_{13} = f_{31} = 0$ it is only necessary to find the flexibility terms f_{11}, f_{22}, f_{33} and f_{23}. In the following, functions of x along the beam are written such that $x = 0$ is located at the start node and $x = L$ is at the end node.

To find the flexibility coefficient, f_{11}, a unit real load is placed at degree of freedom 1 in the natural system, from which the axial force in the beam, $N(x) = 1$, is determined as a function of position x along the length of the beam. Next, a unit virtual load is placed at degree of freedom 1 in the natural system, from which the axial force in the beam, $n(x) = 1$, is determined as a function of position x along the length of the beam. Then using the method of virtual work for axial displacements the flexibility coefficient is calculated.

$$f_{11} = \int_0^L \frac{N(x)n(x)}{AE}\, dx = \frac{L}{AE}. \tag{5}$$

To find the flexibility coefficient, f_{22}, a unit real load (unit moment in this case) is placed at degree of freedom 2 in the natural system, from which the moment in the beam, $M(x) = x/L - 1$, is determined as a function of position x along the length of the beam. Next, a unit virtual load (unit moment in this case) is placed at degree of freedom 2 in the natural system, from which the moment in the beam, $m(x) = x/L - 1$, is determined as a function of position x along the length of the beam. Then using the method of virtual work for flexural displacements the flexibility coefficient is calculated.

$$f_{22} = \int_0^L \frac{M(x)m(x)}{EI}\,dx = \int_0^L \frac{(x/L-1)^2}{EI}\,dx = \frac{L}{3EI}. \tag{6}$$

To find the flexibility coefficient, f_{33}, a unit real load (unit moment in this case) is placed at degree of freedom 3 in the natural system, from which the moment in the beam, $M(x) = x/L$, is determined as a function of position x along the length of the beam. Next, a unit virtual load (unit moment in this case) is placed at degree of freedom 3 in the natural system, from which the moment in the beam, $m(x) = x/L$, is determined as a function of position x along the length of the beam. Then using the method of virtual work for flexural displacements the flexibility coefficient is calculated.

$$f_{33} = \int_0^L \frac{M(x)m(x)}{EI}\,dx = \int_0^L \frac{(x/L)^2}{EI}\,dx = \frac{L}{3EI}. \tag{7}$$

To find the flexibility coefficient, f_{23}, a unit real load (unit moment in this case) is placed at degree of freedom 3 in the natural system, from which the moment in the beam, $M(x) = x/L$, is determined as a function of position x along the length of the beam. Next, a unit virtual load (unit moment in this case) is placed at degree of freedom 2 in the natural system, from which the moment in the beam, $m(x) = x/L - 1$, is determined as a function of position x along the length of the beam. Then using the method of virtual work for flexural displacements the flexibility coefficient is calculated.

$$f_{23} = \int_0^L \frac{M(x)m(x)}{EI}\,dx = \int_0^L \frac{(x/L)(x/L-1)}{EI}\,dx = \frac{-L}{6EI}. \tag{8}$$

For the above examples the beam element is uniform in cross-section and therefore the area, A, and moment of inertia , I, are constant. The flexibility matrix for this case is

$$\mathbf{D}_n = \begin{bmatrix} \frac{L}{AE} & 0 & 0 \\ 0 & \frac{L}{3EI} & \frac{-L}{6EI} \\ 0 & \frac{-L}{6EI} & \frac{L}{3EI} \end{bmatrix}. \tag{9}$$

5 The equivalence of virtual work in the complete and natural systems

The purpose of this section is to demonstrate that there is a relationship between the stiffness matrix in the natural system and the complete system. This relationship is found by recognizing the equivalence of virtual work in the two systems.

It is mentioned previously that for the purposes herein the complete and natural systems are tied together such that the bending and axial deformations are equivalent in both systems. From this requirement it follows that virtual work must be equivalent in the natural and complete systems. This equivalence is expressed as follows:

$$\mathbf{f}_n^T \boldsymbol{\delta}_n = \mathbf{f}^T \boldsymbol{\delta}, \tag{10}$$

where $\mathbf{f}_n$ is a column vector of forces associated with the degrees of freedom in the natural system and $\boldsymbol{\delta}_n$ is a column vector of virtual displacements associated with the natural degrees of freedom. The force and virtual displacement column vectors $\mathbf{f}$ and $\boldsymbol{\delta}$ are associated with the degrees of freedom in the complete system. By kinematic arguments the relationship between complete and natural degrees of freedom is expressed in equation (1). Therefore, in terms of virtual displacements

$$\boldsymbol{\delta}_n = \boldsymbol{\Gamma}\boldsymbol{\delta}. \tag{11}$$

Furthermore, the relations of force, stiffness and displacement are known as

$$\mathbf{f} = \mathbf{k}\boldsymbol{\delta} \quad \text{and} \quad \mathbf{f}_{\mathrm{n}} = \mathbf{k}_{\mathrm{n}}\boldsymbol{\delta}_{\mathrm{n}}. \tag{12}$$

By substituting (12) into (10) gives

$$(\mathbf{k}_n\boldsymbol{\delta}_n)^T\boldsymbol{\delta}_n = (\mathbf{k}\boldsymbol{\delta})^T\boldsymbol{\delta}. \tag{13}$$

Upon using the transpose operator and the symmetry of the stiffness matrices

$$\boldsymbol{\delta}_n^T\mathbf{k}_n\boldsymbol{\delta}_n = \boldsymbol{\delta}^T\mathbf{k}\boldsymbol{\delta}. \tag{14}$$

Next, substitute (11) into (14), which yields

$$(\boldsymbol{\Gamma}\boldsymbol{\delta})^T\mathbf{k}_n(\boldsymbol{\Gamma}\boldsymbol{\delta}) = \boldsymbol{\delta}^T\mathbf{k}\boldsymbol{\delta}. \tag{15}$$

Upon using the transpose operator the following is obtained

$$\boldsymbol{\delta}^T(\boldsymbol{\Gamma}^T\mathbf{k}_n\boldsymbol{\Gamma})\boldsymbol{\delta} = \boldsymbol{\delta}^T\mathbf{k}\boldsymbol{\delta}. \tag{16}$$

From (16) it is evident that

$$\mathbf{k} = \boldsymbol{\Gamma}^T\mathbf{k}_n\boldsymbol{\Gamma}, \tag{17}$$

which was the objective to show. The expression on the right side of equation (17) is sometimes referred to as a congruent transformation.

6 Constructing the stiffness matrix of a uniform beam

The stiffness matrix for a uniform beam element is now easily found by using the result of the previous section and recognizing that the stiffness matrix in the natural degree of freedom system is simply the inverse of the flexibility matrix found previously. These observations yield the stiffness matrix for a 2D beam element in local coordinates (complete degrees of freedom). That is

$$\mathbf{k} = \boldsymbol{\Gamma}^T\mathbf{k}_n\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^T(\mathbf{D}_n^{-1})\boldsymbol{\Gamma} \tag{18}$$

By symbolically performing these operations using the results given in (2) and (9) the standard stiffness matrix for a 2D beam element is obtained as

$$\mathbf{k} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}. \tag{19}$$

7 Extension to the case of a tapered beam of I-shaped cross-section.

It is possible to extend the results of the previous section to the case of an I-beam (or any beam for that matter) of non-uniform cross-section. In particular the case of a linear varying cross-section is illustrated below in Figure 5. The only change to the procedure already presented is the evaluation of the flexibility coefficients f_{11}, f_{22}, f_{33} and f_{23}. For example, if a linear varying I-beam cross-section is used, the area , $A(x)$, is now a function of x. Also, the moment of inertia, $I(x)$, is a function of x. The resulting integrals for the flexibility coefficients are difficult to evaluate in closed form and so the integrals are numerically evaluated. The axial flexibility coefficient takes the form

$$f_{11} = \int_0^L \frac{N(x)n(x)}{A(x)E}\, dx \tag{20}$$

and a typical flexibility coefficient takes the form

$$f_{ij} = \int_0^L \frac{M_j(x)m_i(x)}{EI(x)}\, dx. \tag{21}$$

Hence, the flexibility matrix, $\mathbf{D}_n$, is constructed numerically and inverted to obtain $\mathbf{k}_n$. Then by the congruent transformation using $\mathbf{\Gamma}$, the complete stiffness matrix is obtained.

For the case of Figure 5 the depth of beam web as a function of x is

$$d(x) = d_1 + (d_2 - d_1)\frac{x}{L}, \tag{22}$$

the area is

$$A(x) = 2t_f b_f + t_w d(x) \tag{23}$$

and the moment of inertia is

$$I(x) = t_w \frac{d(x)^3}{12} + 2t_f b_f \left(\frac{(d(x) + t_f)}{2}\right)^2 + \frac{2b_f t_f^3}{12}. \tag{24}$$

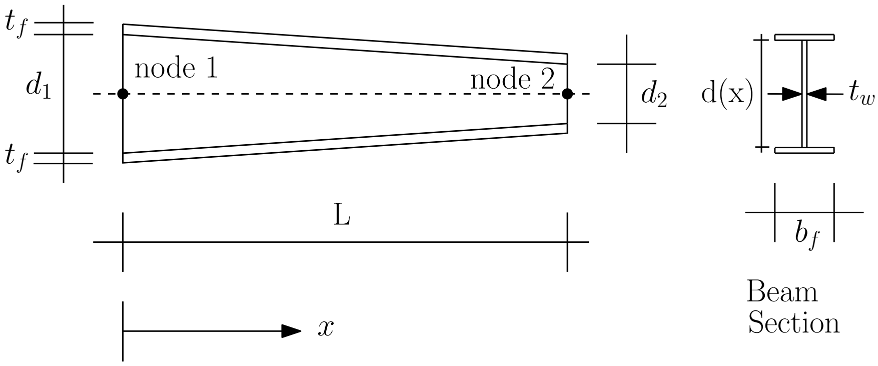

Figure 5: Linear Tapered I-beam.

8 Example results for tapered I-beam cantilever.

Consider the case of a tapered cantilever I-beam fixed at node 1 of Figure 5. The beam has the following properties, $d_1 = 10$ inches, $d_2 = 2$ inches, $t_f = 0.5$ inches, $t_w = 0.5$ inches, $b_f = 4$ inches, $L = 120$ inches. The modulus of elasticity for the beam is $E = 29000$ ksi. A downward acting point load of 10 kips is placed at node 2. If the reader tries to implement the above procedures in a frame analysis program the following results for this tapered cantilever I-beam may be used as a check for correctness of results. The results are obtained using a frame analysis program written in Matlab.

The result for the flexibility matrix in natural coordinates is (units of kips and inches used throughout)

$$\mathbf{D}_n = \begin{bmatrix} +6.08055e-4 & +0.00000000 & +0.00000000 \\ +0.00000000 & +1.75211e-5 & -1.83650e-5 \\ +0.00000000 & -1.83650e-5 & +8.37943e-5 \end{bmatrix}. \tag{25}$$

Once the above matrix is obtained it is a simple task to find the inverse, to obtain $\mathbf{k}_n$, and then by the congruent transformation, equation (18), obtain the complete stiffness matrix for the beam element. The resulting complete stiffness matrix for the beam element is

$$\mathbf{k} = \begin{bmatrix} 1644.6 & 0 & 0 & -1644.6 & 0 & 0 \\ 0 & 8.4769 & 752.79 & 0 & -8.4769 & 264.44 \\ 0 & 752.79 & 74096 & 0 & -752.79 & 16239 \\ -1644.6 & 0 & 0 & 1644.6 & 0 & 0 \\ 0 & -8.4769 & -752.79 & 0 & 8.4769 & -264.44 \\ 0 & 264.44 & 16239 & 0 & -264.44 & 15493 \end{bmatrix}. \tag{26}$$

The reactions at the fixed support are Axial = 0.0, Shear = 10 kips upwards, Moment = 1200 k-in counter-clockwise. The tip displacement of the right end of the cantilever is $\delta_t = -2.523$ inches and the end rotation is $\theta = -0.043063$ radians.

A check on these results may be obtained by using a standard frame analysis program and breaking up the tapered beam into smaller pieces of equal length as shown in Figure 6. The section properties for each of these pieces are taken from the point on the true tapered beam that corresponds to the midpoint of a given piece. Using this procedure the tip deflection

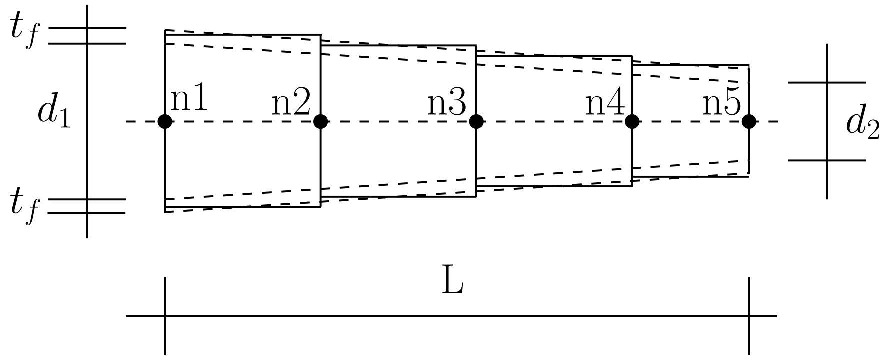

Figure 6: Example: Linear tapered I-beam broken up into 4 smaller uniform segments.

Segments, n	δ (in)	% Error=$\left(\frac{\delta-\delta_t}{\delta_t}\right)$ 100%
1	-3.8692	53.36
2	-2.9349	16.33
3	-2.7104	7.43
4	-2.6280	4.16
5	-2.5897	2.64
6	-2.5690	1.82
10	-2.5392	0.64
20	-2.5269	0.15
50	-2.5235	0.02

Table 1: Tip displacement error for number of segments used to approximate tapered cantilever.

results are compared to various cases with number of segments n. Table 1 illustrates the convergence as the number of segments n increases. These results illustrate the benefit of using a true tapered I-beam formulation. At least four segments are necessary to obtain a deflection error below 5%.

9 Conclusion

A general procedure for finding the stiffness matrix of a 2D beam element is provided. The procedure is demonstrated for the case of a beam of uniform cross-section along its length. The procedure is also extended to the case of a beam with a linearly tapered I-shaped cross-section. It is straight forward to extend the procedure to cases of nonlinear cross-section variation. The motivation for this work is the dissemination of information that is known in the structural mechanics community, but is sometimes difficult to find in this carefully explained form. Furthermore, this information has direct application in the analysis of premanufactured steel buildings, where it is common for such buildings to be fabricated with linearly tapered I-beam sections for beams and columns of portal frames. In addition, in other areas of engineering, such as the aircraft industry, nonuniform beams are very common and a general approach for generating the stiffness matrix is useful.

10 References

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