Moment Distribution
The Real Explanation, And Why It Works

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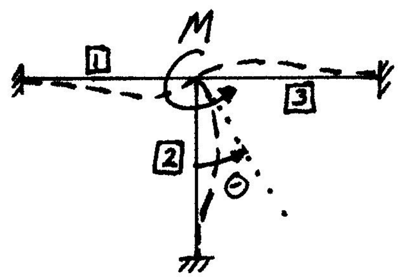
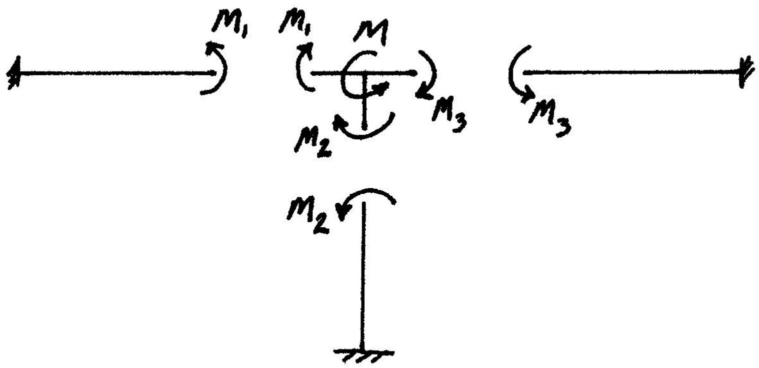
To develop an explanation of moment distribution and why it works, we first need to develop some *tools*. These tools, concepts or pieces of information will be helpful to us as we create the explanation of moment distribution and why it works.

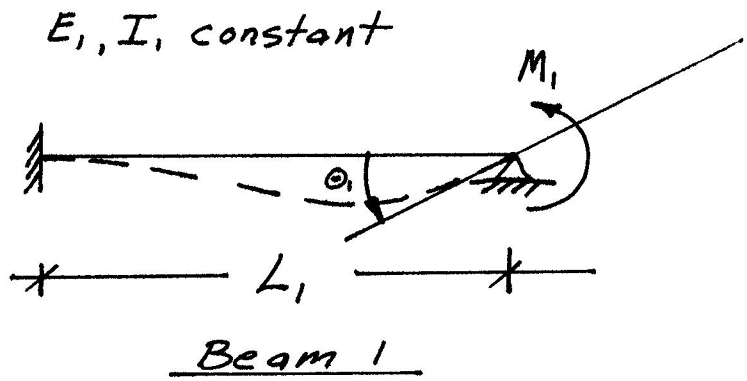
Tool 1 - Distribution Factor. Consider the following structure with an applied external moment at the joint intersected by all three beam elements.



When M is applied, the joint deforms by a rotation θ. All three members undergo the deformation θ. Hence due to compatibility of rotation at the joint we may say that the individual beam rotations at the joint are equivalent, that is $\theta_1 = \theta_2 = \theta_3 = \theta$. Now let us draw a free body diagram of the joint and we see that the beams work together to resist the moment M.



Hence we know from moment equilibrium that $M = M_1 + M_2 + M_3$. Now, if we look at the FBD of the beam 1 we note that it can rotate at the right end, but not translate (ie, it cannot move in the x or y direction if we ignore axial deformations), hence we may model beam 1 as follows:



Hence since beam 1 is acting like a spring resisting rotation we may say that $M_1 = k_1\theta_1$, where k_1 is the beams rotational spring stiffness. (When the rotation at the right end of the beam is $\theta_1 = 1\ rad$ then $M_1 = 4E_1I_1/L_1$.) Similarly we may model beams 2 and 3, so that $M_2 = k_2\theta_2$, $M_3 = k_3\theta_3$. We may also model the whole structural system of our original structure as $M = k\theta$, where k is the total rotational spring stiffness of the joint with the three intersecting beams. Thus recalling our compatibility equation:

$$\theta_1 = \theta_2 = \theta_3 = \theta \tag{1}$$

we may write that

$$\frac{M_1}{k_1} = \frac{M_2}{k_2} = \frac{M_3}{k_3} = \frac{M}{k} \tag{2}$$

Hence in terms of the total moment we may express all three beam moments (M_i) as

$$M_1 = \frac{k_1}{k}M,\ M_2 = \frac{k_2}{k}M,\ M_3 = \frac{k_3}{k}M. \tag{3}$$

Considering now the moment equilibrium of the joint we see that

$$k_1\theta_1 + k_2\theta_2 + k_3\theta_3 = k\theta \tag{4}$$

,but due to compatibility we see that since $\theta_1 = \theta_2 = \theta_3 = \theta$ we have

$$(k_1 + k_2 + k_3)\theta = k\theta \tag{5}$$

or that $k = k_1 + k_2 + k_3 = \Sigma k_i$, where $i = 1, 2, 3$.

Finally, we rewrite the expressions for M_i as

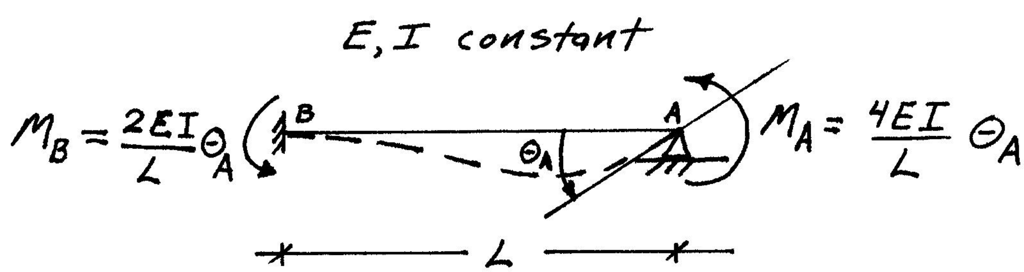
$$M_1 = \frac{k_1}{\Sigma k_i}M,\ M_2 = \frac{k_2}{\Sigma k_i}M,\ M_3 = \frac{k_3}{\Sigma k_i}M. \tag{6}$$

The expressions $\frac{k_j}{\Sigma k_i}$ are called distribution factors. They describe how to distribute the external moment applied at a joint to the individual beams intersecting the joint. Further, we observe that these distribution factors may be interpreted as saying that the individual beams intersecting at a joint resist moment in proportion to their stiffness. Hence the most stiff beam will resist the most moment. Also, we note that the sum of the distribution factors(DF) at a joint will always equal 1. For our example

$$DF_1 + DF_2 + DF_3 = \frac{k_1}{k_1+k_2+k_3} + \frac{k_2}{k_1+k_2+k_3} + \frac{k_3}{k_1+k_2+k_3} = 1 \tag{7}$$

This completes Tool 1. You should understand that an external moment applied at a joint of intersecting beams is distributed to each beam in proportion to each beam's rotational stiffness relative to the total rotational stiffness of all of the intersecting beams. This distribution may be accomplished using the distribution factors.

Tool 2 - Carry Over Factors. Let us examine more carefully the free body diagram idealization for one beam cut from the original structure.



M_A, required to cause a rotation, θ_A, may be expressed as follows:

$$M_A = \frac{4EI}{L}\theta_A \tag{8}$$

When this is true we find that

$$M_B = \frac{2EI}{L}\theta_A, \tag{9}$$

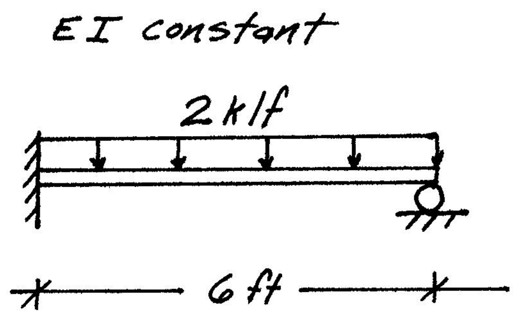
or that $M_B = \frac{1}{2}M_A$. Hence we may say that a moment of magnitude $\frac{1}{2}$ of moment A is 'carried over' to the left end of the beam when M_A is applied at the right. Hence the carry over factor, CO, equals $\frac{1}{2}$ for the structure. Notice also that this is a positive $\frac{1}{2}$ since M_A and M_B are both rotating in the same direction.

This completes Tool 2. You should understand that when a moment is applied at the pinned end of the structure above, a moment of magnitude one half of that applied moment is carried over to the fixed end of the structure. This implies a carry over factor of $+\frac{1}{2}$ from point A to point B of the structure.

Tool 3 - Fixed End Moment Tables. Previously determined fixed end moments of beams for various load configurations will be needed to do the moment distribution procedure. A table containing such information is included on the back cover of your text book by Hibbeler.

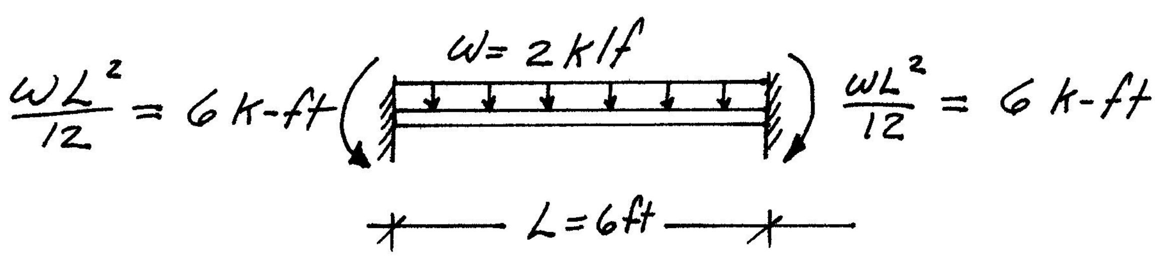
Moment Distribution - Explained. With tools 1, 2 and 3 we are now equipped to understand moment distribution.

Example - 1. Let us first examine a simple example. Consider the following indeterminate structure.

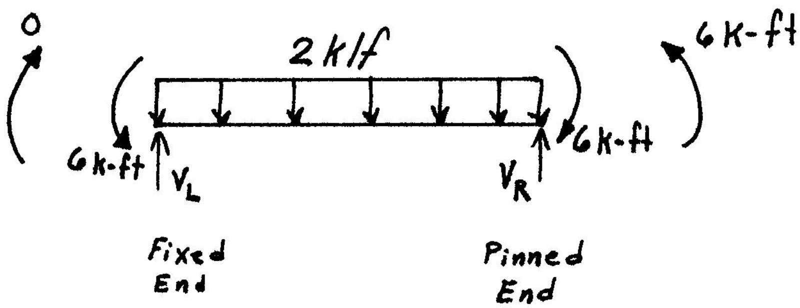


For this indeterminate structure we cannot solve for the reactions by equilibrium alone, hence in this example we resort to the moment distribution technique. In this technique we focus on the internal moments at each joint. At interior joints we know that these moments must be in equilibrium. Our goal is to find the internal moments at the structure joints and by so doing, eliminate enough of the redundants so that the remaining structure reactions may be determined by suitable application of equilibrium equations. Observing how we solve the above simple fixed pinned beam is intended to illustrate the logic behind which moment distribution works.

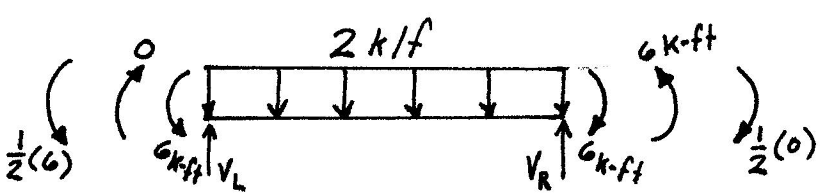
Step 1. First, we convert the structure to a structure for which we know the solution. This can be done by converting the structure to a single span beam with fixed end supports with a distributed loading. From Tool 3 we can calculate the fixed end moments.



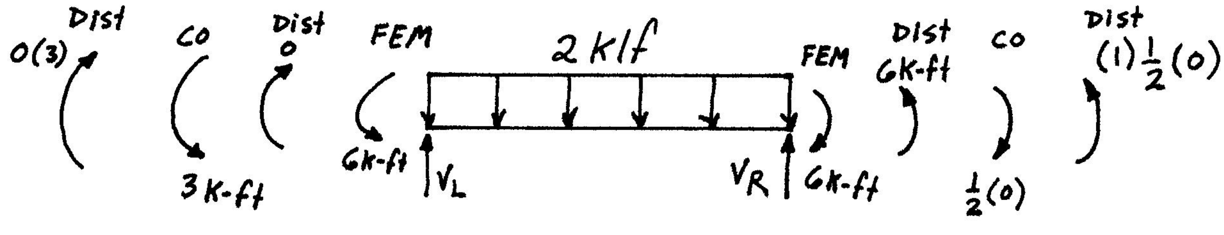
Step 2. The above structure is not the true structure. So, we eliminate the 'false' fixity of the right hand support by applying a counterclockwise moment at the right hand support equal in magnitude to the 'false' clockwise 6 k-ft moment. When we do this we really are distributing the 'correcting' counterclockwise moment to each beam intersecting the joint at the right. But, in this example, there is only one intersecting beam, and that is the beam itself. Hence the distribution factor is DF= $\frac{k_{beam}}{k_{beam}} = 1$. Hence the moment distributed to the beam is 6 k-ft *counterclockwise* (as stated previously to correct the 'false' *clockwise* moment). This step uses Tool 1, which effectively removes the fixity and creates a pin at the right. You might say we have released the fixity we initially imposed on the structure at the right. We also do the same thing at the left, however, here the DF=0 hence the correcting moment is 0, which is what we would expect for a fixed support.



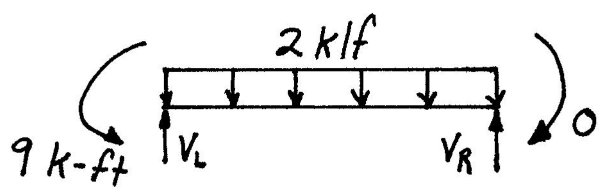
Step 3. Next we recall that when we apply a moment to the right end of a fixed pinned beam a moment of magnitude $\frac{1}{2}$ of the moment applied at the pinned end is caused at the fixed end. This step uses Tool 2, the carry over factor. Hence half of 6 k-ft goes to the left side of the beam. Last, we momentarily imagine the beam as pinned(left) and fixed(right) and thus half of the 0 correcting moment at the fixed support goes over to the right.



Step 4. Next we distribute the carried over moments at each joint just like we distributed the FEM's under step 2. At the right we have DF=1 times zero equals zero, and at the left we have DF=0 times carried over moment=3 k-ft equals zero.



Step 5. Now we sum up all the moments we have imposed on the structure in steps 1 through 4. And we check to see if joint equilibrium has been reached at each joint.



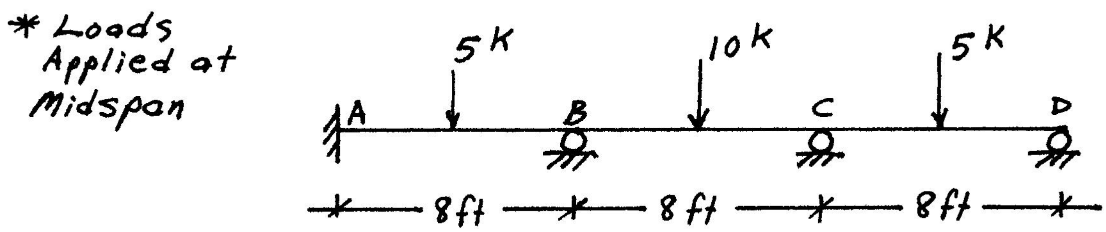
Step	Action	A	B
0	DF	0.0	1.0
1	FEM	-6.0	+6.0
2	DIST	+0.0	-6.0
3	CO	-3.0	+0.0
4	DIST	+0.0	+0.0
5	ΣM(k-ft)	-9.0	+0.0

Summary

1. Create a structure we know the solution for, a 'fixed-fixed' beam.
2. Release the 'false' fixity by adding in an equal but opposite moment at the joint(s), and distribute it to the beam(s) at the joint.
3. Recognize that $\frac{1}{2}$ of the moment(s) are carried over to the fixed support(s).
4. Distribute the carried over moment(s) to the joint(s).
5. Sum up all of the resulting applied moments and see if the results make sense at external support reactions and if we have moment equilibrium at the internal joints (if

we have internal joints in our structure). If equilibrium has not been achieved we may have to repeat steps 3 and 4 more times then do step 5 again and see if equilibrium has been reached.

Example - 2. Now let's look at a more complex example. Consider the following structure for which we shall determine the internal moments at each joint by the moment distribution technique. We shall assume that EI are constant and the same for all beams.



Step	Action	A	B	B	C	C	D
1	DF	0.0	0.5	0.5	0.5	0.5	1.0
2	FEM	-5.0	+5.0	-10.0	+10	-5.0	+5.0
3	DIST	0.0	2.5	2.5	-2.5	-2.5	-5.0
4	CO	1.25	0.0	-1.25	1.25	-2.5	-1.25
5	DIST	0.0	0.625	0.625	0.625	0.625	1.25
6	CO	0.3125	0.0	0.3125	0.3125	0.625	0.3125
7	DIST	0.0	-0.1563	-0.1563	-0.4688	-0.4688	-0.3125
8	ΣM(k-ft)	-3.44	7.97	-7.97	9.22	-9.22	0.0
9	'Exact' M(k-ft)	-3.46	8.08	-8.08	9.23	-9.23	0.0
10	% error	0.62	1.32	1.32	0.12	0.12	0.00

Explanation

1. Determine distribution factors for the beams at joints A, B, C and D. These factors will be used in steps 3, 5 and 7 whenever we *distribute* moments at a joint.

2. Fix all joints, to create a structure of fixed end moment beams. A 'false' structure we can solve is the result. FEM's (AB and CD) equal $\frac{5(8)}{8} = 5$ k-ft. FEM (BC) equals $\frac{10(8)}{8} = 10$ k-ft. Choose clockwise moments as positive.

3. The fixed fixed beam structure is not the real situation, so we release each joint one at a time and put in a moment to cancel the sum of the fictitious moments at a joint and when we do so, we distribute the cancellation moments in accordance with the distribution factors. While one joint is released the others are fixed still. However, in our table, every time we do distribution, we do the distribution process simultaneously for all joints in the current row of the table above(this occurs at some point in the process in rows 3, 5 and 7 of the table).

4. While one joint is fixed and the other released, $\frac{1}{2}$ of the cancellation moment goes to the opposite side of the beam span in accordance with a $\frac{1}{2}$ CO factor.

5. Repeat step 3, but now the sum of carry over moments at each joint are now the false fixed moments which we cancel out.
6. Carry over moments as in step 4.
7. Repeat step 5.
8. Finally, sum all moments and see if internal equilibrium has been achieved at each joint. In this example we see that equilibrium has been achieved, hence further iterations are unnecessary. If equilibrium had not been achieved further iterations of CO, Dist, CO, Dist ... would have been necessary.
9. For comparison, moments determined by an 'exact' computer analysis are given.
10. The percent error between the moments determined by the moment distribution process and the computer analysis moments is quite small and well within engineering accuracy necessary for structural design purposes.

Conclusion. This is the process of moment distribution. It is a process of creating a structure we can solve, then correcting it to get the real structure by successively adding in moment corrections and locking and unlocking joints. By so doing we let moments shift around until a convergence to joint equilibrium is achieved. In the example just given we could have done several more iterations and perhaps improved the accuracy of the final moments calculated. However, after only the three iterations shown the final moments are within 2% , which is well within the accuracy necessary for structural design calculations.