Rigid Diaphragm Analysis
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1 Introduction

A rigid diaphragm analysis is necessary when a roof or floor diaphragm, defined as being rigid, is subjected to lateral seismic forces. A roof or a floor made of a concrete slab or concrete on metal decking is an example of a rigid diaphragm. In contrast, flexible diaphragms are usually comprised of large areas of plywood sheathing or light gauge metal roofing.

Rigid diaphragms are laterally supported by walls (or some other type of lateral resisting element). Seismic forces are transfered from the diaphragm to the walls. The goal of a rigid diaphragm analysis is to determine the maximum shear force transferred to each wall by the diaphragm. A number of concepts are necessary to understand a rigid diaphragm analysis. The needed concepts are presented in the following sections.

2 A Building Example

The plan view of a building is shown in Figure 1. The building shown has shear walls indicated. Assume the roof of this building is a solid concrete (slab) rigid diaphragm of constant thickness. The shear walls support the roof diaphragm laterally. Note that the diaphragm has a weight and the walls have weights tributary to the diaphragm also. That is, half the wall height has weight tributary to the roof diaphragm when considering lateral loads resulting from seismic forces. This building example and related calculations are referred to throughout the remaining sections. Unless noted differently distances are given in feet and forces are in kips.

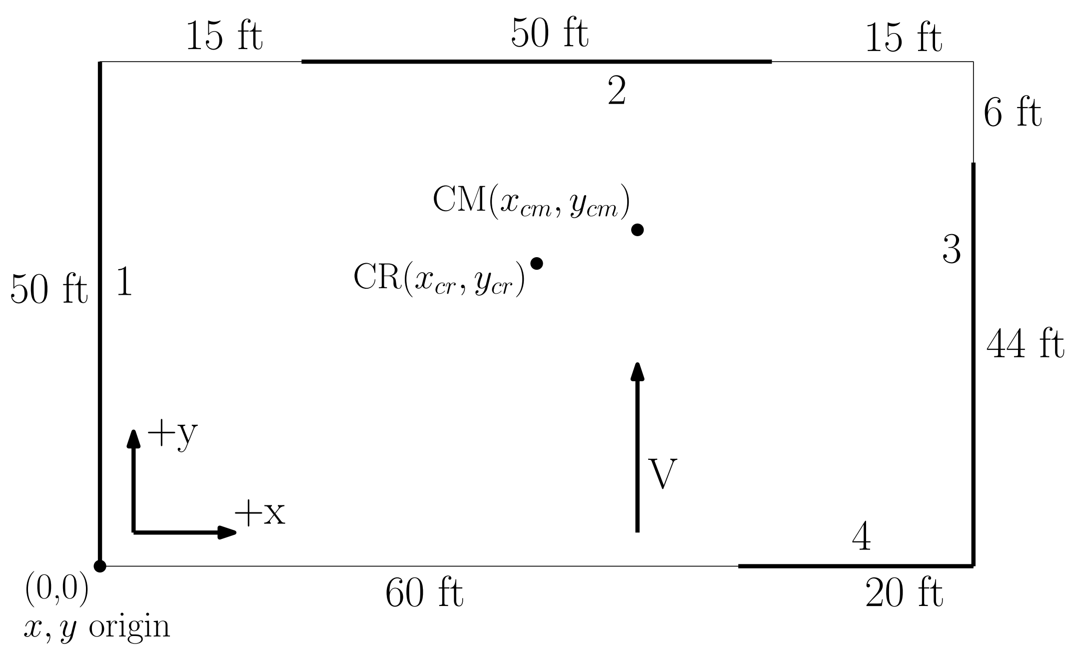

Figure 1: Building Plan View With Shear Walls

k_{th} element of wt.	w_k (kips)	x_k (ft)	y_k (ft)	$w_k x_k$	$w_k y_k$
1 (wall 1)	22.5	0	25	0	562.5
2 (wall 2)	22.5	40	50	900	1125
3 (wall 3)	19.8	80	22	1584	435.6
4 (wall 4)	9.0	70	0	630	0
5 (diaphragm)	200	40	25	8000	5000
	$\sum w_k = 273.8$			$\sum w_k x_k = 11114$	$\sum w_k y_k = 7123.1$

Table 1: Calculations Needed To Determine Center of Mass

The building given in Figure 1 is assumed to have the following characteristics:

- The roof diaphragm is a 4 inch thick concrete slab
- The concrete walls are 12 feet tall and 6 inches thick
- For the purposes of the example assume that the calculated seismic coefficient is, $C_s = 0.3$
- Note that in Figure 1 the seismic shear, V, is acting in the y direction. Earthquakes can act in this direction or on a separate occasion the earthquake could act along the x direction. Hence, the rigid diaphragm analysis must check both of these directions. From these two cases the maximum shear for each wall is used for design. This observation is indicated throughout this document as needed.
- **Note calculations for this example have been carried out with many digits. This is not necessary for sufficient engineering accuracy. However, for purposes of checking computer generated results more digits are sometimes helpful and this document can serve this purpose for someone automating the rigid diaphragm process. At the end the final results are summarized with 3 significant digits.

3 Center of Mass

To do a rigid diaphragm analysis it is necessary to find the center of mass due to all masses that are tributary to the rigid diaphragm. The diaphragm mass itself is included in this analysis. Although this diaphragm characteristic is referred to as center of mass the calculations are actually for the center of weight.

To find the center of mass define the origin (0,0) in the lower left corner of the building. From the origin find (x_k, y_k) to the center of mass of each element of weight that is tributary to the diaphragm. See Table 1 for calculations.

The center of mass coordinates for the rigid diaphragm analysis are then calculated as

$$x_{cm} = \frac{\sum w_k x_k}{\sum w_k} = 40.592 ft \tag{1}$$

and

$$y_{cm} = \frac{\sum w_k y_k}{\sum w_k} = 26.016 ft. \tag{2}$$

4 Rigidity of Walls

Rigidity is another name for stiffness. Each shear wall rigidity must be determined. In practice it is sufficient to find *relative* rigidities for all the walls. For example, for wall i the relative rigidity is $R_i = 1/\Delta_i$. To calculate relative rigidity for walls, Δ_i is taken as

$$\Delta_i = 4\left(\frac{h_i}{L_i}\right)^3 + 3\left(\frac{h_i}{L_i}\right), \tag{3}$$

where h_i is the height of the wall and L_i is the length of the wall. Equation (3) also assumes all walls have the same thickness. If this is not the case then (3) would need to be divided by the correct thickness for each

k_{th} wall	h_k	L_k	Δ_k	R_k
1 (wall 1)	12	50	0.7753	1.2898
2 (wall 2)	12	50	0.7753	1.2898
3 (wall 3)	12	44	0.8993	1.1119
4 (wall 4)	12	20	2.6640	0.3754

Table 2: Relative Rigidity Calculations

wall to obtain the relative rigidities. For the example problem all walls have the same thickness. Hence, (3) is used directly to obtain the relative rigidities for each wall as demonstrated in Table 2.

5 Center of Rigidity

Seismic forces act on the diaphragm. The diaphragm is supported laterally by the walls. Each wall has a rigidity. How rigid the walls are and how they are arranged in the building is used to determine the center of rigidity. Each wall has rigidity when loads act along the length of the wall. Each wall is assumed to have zero rigidity when loads act perpendicular to the wall.

The rigidity for each wall is determined. Table 2 is converted into an R_{kx} and R_{ky} rigidity for each wall. With these rigidities in hand, the center of rigidity is found by using Table 3.

k_{th} wall	R_{kx}	R_{ky}	x_k	y_k	$R_{ky}x_k$	$R_{kx}y_k$
1	0.0	1.2898	0	25	0	0
2	1.2898	0.0	40	50	0	64.492
3	0.0	1.1119	80	22	88.956	0
4	0.3754	0.0	70	0	0	0
	$\sum R_{kx} = 1.6652$	$\sum R_{ky} = 2.4018$			$\sum R_{ky}x_k = 88.956$	$\sum R_{kx}y_k = 64.492$

Table 3: Calculations Needed to Determine the Center of Rigidity

The center of rigidity coordinates for the rigid diaphragm analysis are then calculated as

$$x_{cr} = \frac{\sum R_{ky}x_k}{\sum R_{ky}} = 37.037ft \tag{4}$$

and

$$y_{cr} = \frac{\sum R_{kx}y_k}{\sum R_{kx}} = 38.729ft. \tag{5}$$

6 The Polar Moment of Inertia, J_p

The polar moment of inertia is the rotational moment of inertia determined based on the layout and relative rigidities of the shear walls. The polar moment of inertia is like the J used in mechanics of materials for torsional stress or torsional stiffness. To calculate this quantity the center of rigidity is taken as the origin of a new set of coordinates $(\bar{x}, \bar{y})$. That is

$$\begin{aligned} \bar{x}_k &= x_k - x_{cr} \\ \bar{y}_k &= y_k - y_{cr} \end{aligned} \tag{6}$$

With the above new coordinates, (6), in hand, Table 4 illustrates how to calculate the quantities needed to get J_p.

$$J_p = \sum R_{ky}\bar{x}_k^2 + \sum R_{kx}\bar{y}_k^2 = 4548.7 \tag{7}$$

k_{th} wall	R_{kx}	R_{ky}	$\bar{x}_k$	$\bar{y}_k$	$R_{ky}\bar{x}_k^2$	$R_{kx}\bar{y}_k^2$
1	0	1.2898	-37.0375	-13.7289	1769.355	0.0
2	1.2898	0	2.9625	11.2711	0.0	163.8583
3	0	1.1119	42.9625	-16.7289	2052.408	0.0
4	0.3754	0	32.9625	-38.7289	0.0	563.0346
					$\sum R_{ky}\bar{x}_k^2 = 3821.763$	$\sum R_{kx}\bar{y}_k^2 = 726.8929$

Table 4: Polar Moment of Inertia

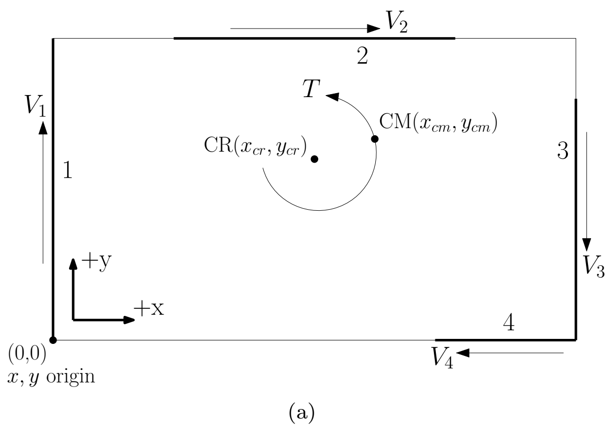

Figure 2: Eccentric Shear Replaced With Statically Equivalent, T and V_y: (a) Torsion About C.R., (b) Direct Shear Acting Through C.R.

7 Torsional Shears

The seismic shear force, V, acts through the center of mass (see Figure 1). Since the center of mass does not coincide with the center of rigidity, the seismic shear force causes torsion. The shear walls resist the torsion as shown in Figure 2a. Hence, shear forces develop in the shear walls due to torsion. Note that the torsion plus direct shear of Figures 2a and 2b are statically equivalent to Figure 1. To simplify the theoretical presentation the following definitions are set forth:

C.R.= center of rigidity
d_i = is the distance from C.R. perpendicular to wall i, that is, d_i is $\bar{x}_i$ or $\bar{y}_i$ depending on the wall orientation
R_i = wall i rigidity, it is R_{ix} or R_{iy} depending on wall orientation. Note that the rigidity is nonzero along the wall, the rigidity is zero perpendicular to the wall, and the wall itself is assumed to have no torsional rigidity about its own centroid. Note also that for now R_i is referred to as the actual rigidity rather than the relative rigidity.
T = Ve = seismic shear times eccentricity = torsion applied to the diaphragm, when V is due to a seismic shear acting in the y direction it is called V_y and a separate loading case V_x when it is acting in the x direction, since earthquakes can act along either coordinate direction of the building
V_i = the shear in wall i reacting to the torsion, T
n = the number of walls resisting the torsion

Torsional equilibrium is required about the C.R. The result is

$$T - \sum_{i=1}^{n} V_i d_i = 0 \quad \Rightarrow \quad T = \sum_{i=1}^{n} V_i d_i. \tag{8}$$

Note that rigidity is another name for stiffness. As a result

$$V_i = R_i \Delta_i. \tag{9}$$

In equation (9) Δ_i is the displacement of wall i due to the applied torsion. Furthermore, a key assumption is that the wall displacement is proportional to distance d_i from the C.R (similar to the elastic vector analysis [4]). This assumption leads to the following:

$$\begin{aligned} &\Rightarrow \Delta_i \propto d_i \\ &\Rightarrow \Delta_i = c d_i \\ &\Rightarrow \frac{V_i}{R_i} = c d_i \\ &\Rightarrow c = \frac{V_i}{R_i d_i} \end{aligned} \tag{10}$$

In equation (10) c is a constant of proportionality. This constant exists based on the assumption that wall displacement is proportional to distance d_i from the C.R. As a result of the last relation

$$\frac{V_1}{R_1 d_1} = \frac{V_2}{R_2 d_2} = \ldots \frac{V_n}{R_n d_n}. \tag{11}$$

Now solve for each of the wall shears in terms of the shear in wall 1. This yields

$$V_1 = \frac{V_1 R_1 d_1}{R_1 d_1}, \quad V_2 = \frac{V_1 R_2 d_2}{R_1 d_1}, \quad \ldots V_n = \frac{V_1 R_n d_n}{R_1 d_1}. \tag{12}$$

Substitute (12) into equation (8) to get

$$T = \sum_{i=1}^{n} \frac{V_1 R_i d_i^2}{R_1 d_1} = \frac{V_1}{R_1 d_1} \sum_{i=1}^{n} R_i d_i^2. \tag{13}$$

Then solving for V_1 obtain

$$V_1 = \frac{R_1 d_1 T}{\sum R_i d_i^2}. \tag{14}$$

Generalizing the above solution for wall 1 to the solution for wall j and realizing that the denominator is J_p, yields

$$V_j = \frac{R_j d_j T}{\sum R_i d_i^2} = \frac{T R_j d_j}{J_p}. \tag{15}$$

Equation (15) can also be rewritten as x and y wall shears for each wall j.

$$V_{jx} = \frac{T R_{jx} \bar{y}_j}{J_p}, \quad V_{jy} = \frac{T R_{jy} \bar{x}_j}{J_p}. \tag{16}$$

Of course each wall only has an x shear or it only has a y shear depending on its orientation. The other shear direction (perpendicular to the wall) is zero.

8 Direct Shears

Direct shears due to V_y are depicted in Figure 2b. Walls take direct shear in proportion to their rigidity (or relative rigidity). As a result

$$V_j = V \frac{R_j}{\sum_{i \parallel V} R_i}, \tag{17}$$

where here $\sum_{i \parallel V} R_i$ only includes the sum of rigidities for walls parallel to the direction of the applied shear, V. V_j is equal to zero for walls perpendicular to V. Table 5 illustrates the direct shear calculations.

x-Direction Load, $V_x = C_s \sum w_k = 82.14$ kips				y-Direction Load, $V_y = C_s \sum w_k = 82.14$ kips		
j_{th} wall	R_{jx}	V_{jx}		j_{th} wall	R_{jy}	V_{jy}
1	0	0		1	1.2898	44.1118
2	1.2898	63.6238		2	0	0
3	0	0		3	1.1119	38.0282
4	0.3754	18.5162		4	0	0
	$\sum R_{jx} = 1.6652$				$\sum R_{jy} = 2.4018$	

Table 5: Direct Shears for Separate Cases of x or y Loading

9 Real Eccentricity of Diaphragm Shear

The torsion induced on the diaphragm is due to a distance between the line of action of V and the C.R. This distance is referred to as an eccentricity. An analysis of the diaphragm is necessary for the case of V acting in the x direction and a completely separate case for V acting in the y direction. When V is acting in the x direction then the eccentricity is $e_y = y_{cm} - y_{cr} = -12.7132$ ft. When V is acting in the y direction then the eccentricity is $e_x = x_{cm} - x_{cr} = 3.5542$ ft. These real eccentricities occur due to a difference between the C.M. and the C.R.

10 Accidental Eccentricity of Diaphragm Shear

The building code [2] requires accidental eccentricities to be included also. These cause an additional amount of torsion and are required to be 5% of the longest building dimension in each dirction. In this case the accidental eccentricities are

$$\begin{aligned} \hat{e}_x &= 0.05(longest\ x\ bldg\ dimension) = 0.05(80) = 4.0ft, \\ \hat{e}_y &= 0.05(longest\ y\ bldg\ dimension) = 0.05(50) = 2.5ft. \end{aligned} \tag{18}$$

11 Putting It All Together

For each wall three things contribute to the total shear. First, walls parallel to the applied seismic shear have direct shears according to (17). Second, it is standard practice to include real torsional shears according to equations (16). That is, the formulas, as derived, automatically provide the correct sign to the calculated quantity. Last, it is standard practice to add the *absolute* value of accidentally occurring torsional shears according to equations (16). In the following subsections the consequences of these standard practices are evident in the equations, where contributions due to real eccentricities and accidental eccentricities are added separately. This is easy to accomplish because the total torsion is easily broken up as $T = V(e_{real} + e_{accidental}) = Ve_{real} + Ve_{accidental}$.

11.1 Case of Seismic Shear Acting in x Direction

In this case the applied seismic shear is labeled as V_x. The total shear for wall j has the following basic formula

$$V_j = direct\ shear + real\ torsional\ shear + |accidental\ torsional\ shear|. \tag{19}$$

Using equations (16) and (17) for walls oriented in the x direction the result is [1]

$$V_{jx} = V_x \frac{R_{jx}}{\sum R_{ix}} + \frac{V_x e_y R_{jx} \bar{y}_j}{J_p} + \left| \frac{V_x \hat{e}_y R_{jx} \bar{y}_j}{J_p} \right|, \tag{20}$$

x-Direction Load, $V_x = 82.14$ kips, $T = V_x e_y$ or $V_x \hat{e}_y$				
j_{th} wall	$V_{jx(real)}$	$\lvert V_{jx(accidental)} \rvert$	$V_{jy(real)}$	$\lvert V_{jy(accidental)} \rvert$
1	0	0	10.9673	2.1567
2	-3.3375	0.6563	0	0
3	0	0	-10.9673	2.1567
4	3.3375	0.6563	0	0

Table 6: Torsional Shears for Case of x Direction Loading

j_{th} wall	V_{jx}	V_{jy}
1	0	13.1240
2	60.9426	0
3	0	13.1240
4	22.5101	0

Table 7: Total Shears x-direction of loading

where for equation (16) the torsion T has been rewritten as the shear, V_x, times the applicable eccentricity. For walls oriented in the y direction the result is

$$V_{jy} = 0 - \frac{V_x e_y R_{jy} \bar{x}_j}{J_p} + \left| \frac{V_x \hat{e}_y R_{jy} \bar{x}_j}{J_p} \right|. \tag{21}$$

Notice that the walls oriented in the y direction have zero direct shear contribution since the seismic shear force is acting in the x direction. See results in Table 6 for torsional shears. Using equations (20) and (21) the total shears are shown in Table 7. In (21) a negative sign on real torsional shears is needed to correctly report forces as they act on the wall.

11.2 Case of Seismic Shear Acting in y Direction

In this case the applied seismic shear is labeled as V_y. The total shear for wall j has the following basic formula

$$V_j = direct\ shear + real\ torsional\ shear + |accidental\ torsional\ shear|. \tag{22}$$

Using equations (16) and (17) for walls oriented in the y direction the result is

$$V_{jy} = V_y \frac{R_{jy}}{\sum R_{iy}} + \frac{V_y e_x R_{jy} \bar{x}_j}{J_p} + \left| \frac{V_y \hat{e}_x R_{jy} \bar{x}_j}{J_p} \right|, \tag{23}$$

where for equation (16) the torsion T has been rewritten as the shear, V_y, times the applicable eccentricity. For walls oriented in the x direction the result is

$$V_{jx} = 0 - \frac{V_y e_x R_{jx} \bar{y}_j}{J_p} + \left| \frac{V_y \hat{e}_x R_{jx} \bar{y}_j}{J_p} \right|. \tag{24}$$

Notice that the walls oriented in the x direction have zero direct shear contribution since the seismic shear force is acting in the y direction. See results in Table 8 for torsional shears. Using equations (23) and (24) the total shears are shown in Table 9. In (24) a negative sign on real torsional shears is needed to correctly report forces as they act on the wall.

11.3 Wall Shears Used for Design

Each wall j is designed for the worst shear force of the two cases, x seismic loading or y seismic loading. Hence, from Table 7 and Table 9 the maximum possible shear for each wall is summarized in Table 10.

y-Direction Load, $V_y = 82.14$ kips, $T = V_y e_x$ or $V_y \hat{e}_x$				
j_{th} wall	$V_{jx(real)}$	$\lvert V_{jx(accidental)} \rvert$	$V_{jy(real)}$	$\lvert V_{jy(accidental)} \rvert$
1	0	0	-3.0661	3.4507
2	0.9331	1.0501	0	0
3	0	0	3.0661	3.4507
4	-0.9331	1.0501	0	0

Table 8: Torsional Shears for Case of x Direction Loading

j_{th} wall	V_{jx}	V_{jy}
1	0	44.4964
2	1.9832	0
3	0	44.5450
4	1.9832	0

Table 9: Total Shears y-direction of loading

j_{th} wall	$V_{j(max)}$ (kips)
1	44.5
2	60.9
3	44.5
4	22.5

Table 10: Maximum Shears for Design

12 Concluding Comments

1. Shear walls are used in the above presentation. However, the lateral resisting elements, and their associated rigidities, could just as easily have been some other type of lateral resisting element such as braced frames, moment frames or steel shear walls to name a few. Rigidities for any type of lateral resisting element is easily found by applying a 1 kip lateral load to the top of the element. The resulting horizontal displacement, Δ, can be used to calculate the rigidity, $R = 1/\Delta$.

2. In the example, analysis weights tributary to the diaphragm from the shear walls and the diaphragm are included, however, it is likely that in a real life analysis there are many other dead loads that would need to be included. Some examples are columns, mechanical units, beams supporting the diaphragm or ceiling dead loads to name a few.

3. Torsional shears due to accidental torsion have *absolute* values added to the wall shear because the intent of the code is that they always increase direct shears [1].

4. In some implementations of a rigid diaphragm analysis the rigidities are normalized. This is accomplished by first searching through all walls in the building and finding the one with the maximum rigidity. Second, all walls are divided by the maximum rigidity. Hence, after this is done the maximum possible rigidity is 1.0. This process of normalization will of course change the rigidities used throughout the analysis, but also changes the value of J_p from what it would have been if normalization had not been done. The normalization process is not necessary, but sometimes is helpful to make the numbers in calculations easier to work with. The final design shears should come out the same whether normalization is done or not.

5. With the formulas provided it is a straight forward matter to automate a rigid diaphragm analysis in a computer program for any number of shear walls.

6. The present derivation is presented for walls that are oriented only in the x or y directions. It is possible to create a formulation that accounts for walls at an angle to the x axis [3]. However, this case has not been considered herein.

References

[1] *SEAOC Structural/Seismic Design Manual 2009 IBC.* Structural Engineers Association of California, Sacramento, CA, 2012.

[2] ASCE. *Minimum Design Loads for Buildings and Other Structures - ASCE 7-10.* American Society of Civil Engineers, Reston, Virginia, 2010.

[3] Gregg Brandow, Gary Hart, and Ajit Virdee. *Design of Reinforced Masonry Structures.* Concrete Masonry Association of California and Nevada, Citrus Heights, California, 1995.

[4] Charles G. Salmon, Johne E. Johnson, and Faris A. Malhas. *Steel Structures.* Pearson Prentice Hall, Upper Saddle River, New Jersey, 5th edition, 2009.