A Method to Enforce Displacement Boundary Conditions in Finite Element Analysis
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1 Introduction

A simple method to enforce displacement boundary conditions in finite element analysis is presented. The method is effective for any type of finite element analysis such as for trusses, frames, solids, and others. Herein, examples are provided for a single beam element. Although other organizational schemes for displacement boundary condition enforcement are possible, the presented scheme is fairly simple and is quite effective for both linear and nonlinear analysis. The technique illustrated follows a method described by Logan [2].

1.1 A Single 2D Frame Element

Consider the frame element of Figure 1 with six degrees of freedom. For a single frame element the local element forces are related to the local element displacements by the local element stiffness matrix. That is

$$
\left\{
\begin{array}{c}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6
\end{array}
\right\}
=
\left[
\begin{array}{cccccc}
k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\
k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\
k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\
k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\
k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\
k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66}
\end{array}
\right]
\left\{
\begin{array}{c}
d_1 \\
d_2 \\
d_3 \\
d_4 \\
d_5 \\
d_6
\end{array}
\right\},
\tag{1}
$$

or

$$
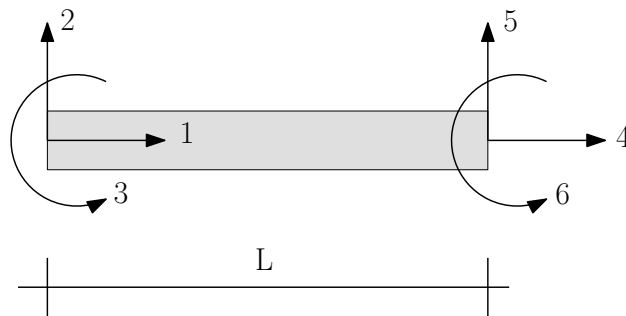\mathbf{f} = \mathbf{k}\mathbf{d}. \tag{2}
$$

Figure 1: Frame element with six degrees of freedom.

1.2   The Global Stiffness

Typically, within a finite element anlaysis for frame elements, the local element stiffness, $\mathbf{k}$, is transformed to a global coordinates element stiffness, $\bar{\mathbf{k}}$, and then *assembled* into the global stiffness matrix, $\mathbf{K}$. For example,

$$\bar{\mathbf{k}} = \mathbf{T}^T \mathbf{k} \mathbf{T} \tag{3}$$

and

$$\mathbf{K} = \underset{i=1}{\overset{n}{\mathsf{A}}}\, \bar{\mathbf{k}}_i, \tag{4}$$

where $\mathbf{T}$ is the standard frame element transformation matrix, A is the assembly operator (see Hughes [1]) and n is the number of elements to be assembled into the global stiffness matrix. Finally, the global system of equations in matrix form is

$$\mathbf{F} = \mathbf{K}\mathbf{u}, \tag{5}$$

where $\mathbf{F}$ is the column vector of externally applied nodal global forces, and u is the column vector of global nodal displacements. With the above definitions (from typical finite element formulations) in place, it is possible to present a method for the enforcement of displacement boundary conditions.

2   Enforcement of Zero Valued Displacement Boundary Conditions

An example is presented with zero valued displacement boundary conditions. To keep the example simple the structure contains a single frame element loaded with a single point load as shown in Figure 2. In this simple case the local element stiffness matrix, transformed element stiffness matrix, and global stiffness matrix coincide. The left end supports cause the frame element to behave like a cantilever beam.
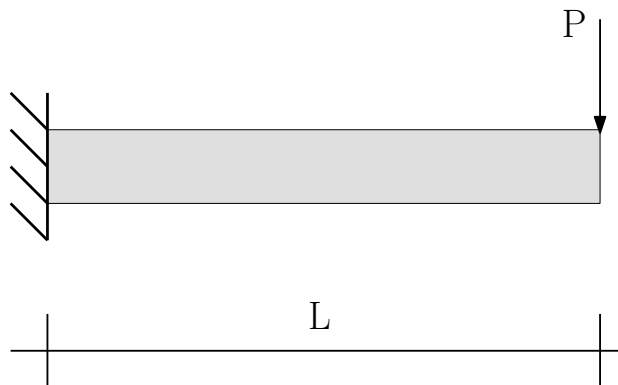
Figure 2: Cantilever frame element with point load at free end.

Algorithmically, there are two steps to enforcing the zero valued displacement boundary conditions.

1. The global force vector, $\mathbf{F}$, is modified by setting each force degree of freedom (dof) i equal to zero that has zero as the specified displacement for dof i.

2. The rows and columns of the global stiffness matrix for each zero valued displacement dof i are set to zero except that position K_{ii} is set equal to 1.0.

In the given example zero valued displacements are at dofs 1,2,3. The results of the prescribed modifications are:

$$\mathbf{F}_s = \left\{ \begin{array}{c} u_1 = 0 \\ u_2 = 0 \\ u_3 = 0 \\ F_4 = 0 \\ F_5 = -P \\ F_6 = 0 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ -P \\ 0 \end{array} \right\} \tag{6}$$

and

$$\mathbf{K}_s = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{44} & K_{45} & K_{46} \\ 0 & 0 & 0 & K_{54} & K_{55} & K_{56} \\ 0 & 0 & 0 & K_{64} & K_{65} & K_{66} \end{array} \right], \tag{7}$$

where $\mathbf{F}_s$ and $\mathbf{K}_s$ are the modified vector and matrix to account for zero valued displacement (support) conditions and the applied global force P. Finally, observe that

$$\mathbf{K}_s \mathbf{u} = \mathbf{F}_s \tag{8}$$

from which the global displacements are obtained as

$$\mathbf{u} = \mathbf{K}_s^{-1} \mathbf{F}_s. \tag{9}$$

The column vector $\mathbf{u}$ gives back the zero displacements at the support dofs 1,2,3 and contains the calculated displacements u_4, u_5, u_6 at the free end of the cantilever due to the applied load P.

Another example with zero valued displacement boundary conditions and applied moment M is shown in Figure 3.
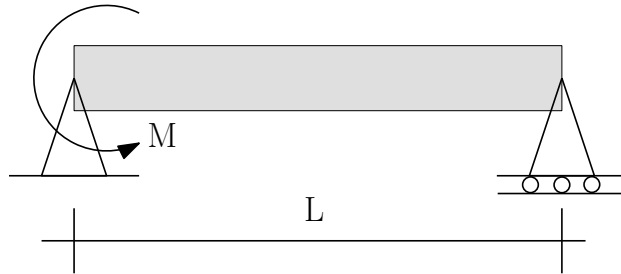

Figure 3: Simply supported frame element with applied moment M.

The modified force vector takes the form

$$
\mathbf{F}_s = \left\{ \begin{array}{c} u_1 = 0 \\ u_2 = 0 \\ F_3 = M \\ F_4 = 0 \\ u_5 = 0 \\ F_6 = 0 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ M \\ 0 \\ 0 \\ 0 \end{array} \right\}
\tag{10}
$$

and the modified stiffness is

$$
\mathbf{K}_s = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{33} & K_{34} & 0 & K_{36} \\ 0 & 0 & K_{43} & K_{44} & 0 & K_{46} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & K_{63} & K_{64} & 0 & K_{66} \end{array} \right].
\tag{11}
$$

3   Enforcement of Nonzero *and* Zero Valued Displacement Boundary Conditions

Demonstration is now made for the case of externally applied loads, zero valued displacements, *and* specified nonzero displacement boundary conditions. Consider the case shown in Figure 4. Specified nonzero displacements are denoted with an overbar $(\bar{u}_4, \bar{u}_6)$.
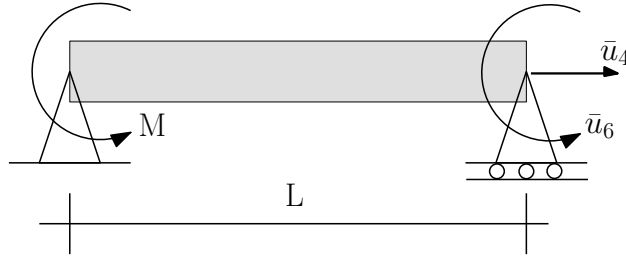

Figure 4: Frame element with nonzero and zero valued specified displacements and applied moment M.

For this type of problem, inside a computer program, it is important to modify the force vector first because its modifications are dependent on the *unmodified* global stiffness matrix. It is helpful to see the matrix terms side by side before any modifications.

$$
\left[\begin{array}{cccccc} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{array} \right], \left\{ \begin{array}{c} u_1 = 0 \\ u_2 = 0 \\ u_3 \\ u_4 = \bar{u}_4 \\ u_5 = 0 \\ u_6 = \bar{u}_6 \end{array} \right\}, \left\{ \begin{array}{c} F_1 \\ F_2 \\ F_3 = M \\ F_4 \\ F_5 \\ F_6 \end{array} \right\}.
\tag{12}
$$

Notice that F_1, F_2, F_4, F_5, and F_6 are unspecified forces because they occur at specified displacement (zero and nonzero) locations, and at such locations nodal forces cannot be simultaneously specified. From (12), the unmodified force vector is

$$\mathbf{F} = \left\{ \begin{array}{c} 0 \\ 0 \\ M \\ 0 \\ 0 \\ 0 \end{array} \right\}. \tag{13}$$

Notice in (13) that dof 3 can be a specified zero or nonzero load. All other force dofs are initially set to zero before $\mathbf{F}$ is modified. In this case the applied moment has been specified at dof 3. If no loads are specified then the structure is only "loaded" by the specified nonzero displacements (times stiffness) as is seen below.

The first step to modify $\mathbf{F}$ is to multiply each row of the global stiffness times $\mathbf{u}$ and set it equal to the corresponding value in (13) to get

$$\begin{aligned}
K_{11}(0) + K_{12}(0) + K_{13}u_3 + K_{14}\bar{u}_4 + K_{15}(0) + K_{16}\bar{u}_6 &= 0 \\
K_{21}(0) + K_{22}(0) + K_{23}u_3 + K_{24}\bar{u}_4 + K_{25}(0) + K_{26}\bar{u}_6 &= 0 \\
K_{31}(0) + K_{32}(0) + K_{33}u_3 + K_{34}\bar{u}_4 + K_{35}(0) + K_{36}\bar{u}_6 &= M \\
K_{41}(0) + K_{42}(0) + K_{43}u_3 + K_{44}\bar{u}_4 + K_{45}(0) + K_{46}\bar{u}_6 &= 0 \\
K_{51}(0) + K_{52}(0) + K_{53}u_3 + K_{54}\bar{u}_4 + K_{55}(0) + K_{56}\bar{u}_6 &= 0 \\
K_{61}(0) + K_{62}(0) + K_{63}u_3 + K_{64}\bar{u}_4 + K_{65}(0) + K_{66}\bar{u}_6 &= 0
\end{aligned} \tag{14}$$

Stiffness terms times nonzero displacements are the only *known* nonzero terms that can be taken to the right hand side of (14). The right hand side becomes a new form for $\mathbf{F}$. The result is

$$\mathbf{F}_{intermediate} = \left\{ \begin{array}{c} -K_{14}\bar{u}_4 - K_{16}\bar{u}_6 \\ -K_{24}\bar{u}_4 - K_{26}\bar{u}_6 \\ M - K_{34}\bar{u}_4 - K_{36}\bar{u}_6 \\ -K_{44}\bar{u}_4 - K_{46}\bar{u}_6 \\ -K_{54}\bar{u}_4 - K_{56}\bar{u}_6 \\ -K_{64}\bar{u}_4 - K_{66}\bar{u}_6 \end{array} \right\}. \tag{15}$$

The force vector $\mathbf{F}_{intermediate}$ in (15) is modified one step further to account for all specified displacements (zero and nonzero). The final modified force vector is

$$\mathbf{F}_s = \left\{ \begin{array}{c} 0 \\ 0 \\ M - K_{34}\bar{u}_4 - K_{36}\bar{u}_6 \\ \bar{u}_4 \\ 0 \\ \bar{u}_6 \end{array} \right\}. \tag{16}$$

It is possible to avoid intermediate equation (15) and go directly from (13) to (16) with appropriate bookkeeping inside a finite element analysis computer program. With $\mathbf{F}_s$ now in

hand the modified global stiffness matrix is created by zeroing each row i and column i for each specified displacement dof i and then setting $K_{ii} = 1.0$. The resulting matrix equations are

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & K_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{Bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6
\end{Bmatrix}
=
\begin{Bmatrix}
0 \\
0 \\
M - K_{34}\bar{u}_4 - K_{36}\bar{u}_6 \\
\bar{u}_4 \\
0 \\
\bar{u}_6
\end{Bmatrix}. \tag{17}
$$

which is

$$
\mathbf{K}_s \mathbf{u} = \mathbf{F}_s. \tag{18}
$$

As in previous examples

$$
\mathbf{u} = \mathbf{K}_s^{-1} \mathbf{F}_s \tag{19}
$$

gives back the correct required specified displacements at dofs 1,2,4,5, and 6 and the unknown displacement at dof 3.

Remarks

1. In the problems the number of specified displacements is similar to or outnumbers the unknown displacements. This may seem inefficient and other schemes for partitioning matrices to reduce the number of equations in (18) may appear to be warranted. However, most problems of interest contain unknown displacements that greatly outnumber the specified displacements. As a result only a small computational savings is generally achieved by reducing the equations (18).

2. It is the position of this article that the bookkeeping and computer implementation of the scheme described herein is easier and more transparent than other solution methods. As a result all dofs are included in the global solution scheme rather than some slightly smaller subset of partitioned matrices.

3. Notice in the last example that if M was zero the force vector would still contain forces by virtue of the stiffness terms times nonzero specified displacments.

4. In finite element literature benchmark problems are sometimes given that require imposition of displacements at mesh nodes instead of forces. The method presented in this article is capable of handling such problems. Furthermore, the method presented is also applicable within the framework of a nonlinear finite element analysis if implemented properly.

4   Conclusion

A method for enforcing displacement boundary conditions within a finite element analysis is presented. Displacement boundary conditions that are numerically zero or numerically nonzero have both been addressed. Other schemes for enforcing boundary equations and solving the finite element equations are possible. However, it is the position of this article

that it is easier in terms of computer program bookkeeping to solve the full set of linear equations.

References

[1] T. J. R. Hughes. *The Finite Element Method - Linear Static and Dynamic Finite Element Analysis.* Dover, Mineola, NY, 1st edition, 2000.

[2] D. L. Logan. *A First Course in the Finite Element Method.* PWS-Kent Publishing Company, Boston, MA, 2nd edition, 1992.