J_2 Plasticity - Derivation
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1 Introduction

It is easy to understand yield as a material failure in a 1D tension test. However, in a 3D state of stress it is less clear when yielding will occur. That is, at a material point in a 3D body it is possible that none of the individual stresses at the point are above yield, however, the *combined effect* of the stresses at the point may be sufficient to cause yielding at the material point. As a result, theories of failure have been hypothesized to obtain criteria for failure (or yielding) for 3D states of stress. J_2 plasticity is the result of one such theory. For ductile metal materials J_2 plasticity does a good job of predicting if a 3D stress state is causing yielding [2, 8]. Clearly, this has important practical applications to engineering design. A derivation for J_2 plasticity is presented in this article. The von Mises yield criterion is another name for J_2 plasticity. This theory arises due to the hypothesis that yielding begins when *distortional strain* energy density reaches a critical (maximum) value [1, 2]. In the ensuing sections the necessary ingredients for a 3D yield criteria (J_2 plasticity) are derived. Finally, the ingredients are put together to provide a formula that predicts if yielding is taking place in a 3D state of stress.

2 Strain Energy Density

Consider a stress block in uniaxial tension as shown in Figure 1. The stress block has differential dimensions, dx, dy, dz, and differential volume $dv = dxdydz$. The block is stressed from 0 to σ_x. Hence, the average stress during loading is $\frac{1}{2}\sigma_x$. The final strain due to loading in the x direction is ε_x. Energy equals work and work equals force times displacement. Therefore, the work performed on the block of material is

$$work = (avg.\ force)(distance) = \frac{1}{2}\sigma_x dzdy(\varepsilon_x dx) = \frac{1}{2}\sigma_x \varepsilon_x dxdydz = \frac{1}{2}\sigma_x \varepsilon_x dv. \tag{1}$$

Strain energy density, w_e, is defined as work per volume.

$$w_e = \frac{work}{volume} = \frac{\frac{1}{2}\sigma_x \varepsilon_x dv}{dv} = \frac{1}{2}\sigma_x \varepsilon_x \tag{2}$$

The result in (2) is for a 1D state of stress. In general, strain energy density for a 3D state of stress, written in tensor (or indicial) notation [3, 6], is

$$w_e = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}, \tag{3}$$

where here, and throughout the remainder of this article, indices range over values from 1 to 3 and the Einstein summation convention is invoked [3, 6].

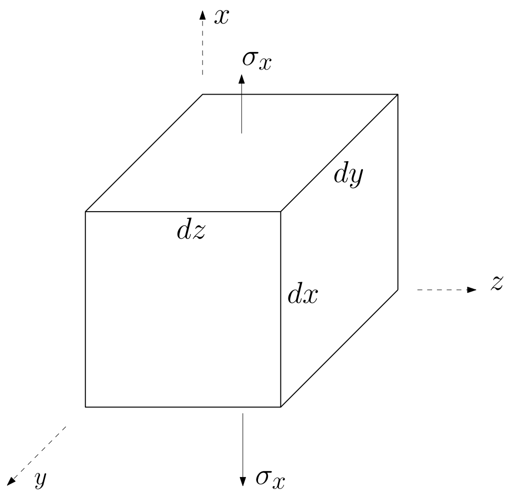

Figure 1: Stress block in uniaxial tension.

3 Stress Strain Relations for Isotropic Elastic Material

For isotropic elastic materials stress is related to strain [7] as follows:

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}, \tag{4}$$

where λ and μ are *Lamé* constants, and δ_{ij} is the kronecker delta.

4 Deviatoric Stress and Volumetric Stress

The stress tensor, $\boldsymbol{\sigma}$, can be decomposed into two parts [7]. Volumetric stresses, $\boldsymbol{\sigma^{vol}}$, produce strains that change the total volume of a given stress block. Deviatoric (or distortional) stresses, $\boldsymbol{\sigma^{dev}}$, do not produce volume change but instead are responsible for strains that cause the stress block to distort or deviate from a cubic or rectangular prism shape.

The volumetric part is found by the following observations. For a principal stress block only volumetric strains are produced. The average stress for this condition is $\frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)$, where $\sigma_1, \sigma_2, \sigma_3$ are the principal stresses. This says that the average stress is one third of the trace of the stress tensor. It follows, since the trace is invariant, that in general the average stress is $\frac{1}{3}(\sigma_{kk})$. As a result the (volumetric) stresses that cause volume change are

$$\sigma_{ij}^{vol} = \frac{1}{3}\sigma_{kk}\delta_{ij}. \tag{5}$$

The deviatoric part is found as follows. First, note that the stress tensor is representable as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\boldsymbol{vol}} + \boldsymbol{\sigma}^{\boldsymbol{dev}}. \tag{6}$$

Hence, from (6), the deviatoric stress tensor is

$$\boldsymbol{\sigma}^{\boldsymbol{dev}} = \boldsymbol{\sigma} - \boldsymbol{\sigma}^{\boldsymbol{vol}}. \tag{7}$$

In indicial notation, using (5) in (7),

$$\sigma_{ij}^{dev} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}. \tag{8}$$

5 Strain Energy Density in Terms of Stresses

The strain energy density expression, (3), is in terms of stresses and strains. For practical applications it is useful to have the strain energy density in terms of stresses only. Hence, use (4) to get

$$\sigma_{ij} - \lambda\delta_{ij}\varepsilon_{kk} = 2\mu\varepsilon_{ij}. \tag{9}$$

Then, from (9), the strains are

$$\varepsilon_{ij} = \frac{1}{2\mu}(\sigma_{ij} - \lambda\delta_{ij}\varepsilon_{kk}). \tag{10}$$

Next, use (4) to obtain

$$\begin{aligned} \sigma_{kk} &= \lambda\delta_{kk}\varepsilon_{mm} + 2\mu\varepsilon_{kk} \\ &= 3\lambda\varepsilon_{mm} + 2\mu\varepsilon_{kk} \\ &= 3\lambda\varepsilon_{kk} + 2\mu\varepsilon_{kk} \\ &= (2\mu + 3\lambda)\varepsilon_{kk}. \end{aligned} \tag{11}$$

Rearranging (11) yields

$$\varepsilon_{kk} = \frac{\sigma_{kk}}{2\mu + 3\lambda}. \tag{12}$$

Using (12) in (10) gives strains in terms of stresses,

$$\varepsilon_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda\delta_{ij}\sigma_{kk}}{2\mu + 3\lambda}\right) = \frac{\sigma_{ij}}{2\mu} - \frac{\lambda\delta_{ij}\sigma_{kk}}{2\mu(2\mu + 3\lambda)}. \tag{13}$$

Last, insert (13) into (3) to get strain energy density in terms of stresses,

$$w_e = \frac{1}{2}\sigma_{ij}\left(\frac{\sigma_{ij}}{2\mu} - \frac{\lambda\delta_{ij}\sigma_{kk}}{2\mu(2\mu + 3\lambda)}\right). \tag{14}$$

6 Strain Energy Density in Terms of Deviatoric Stresses

The strain energy density is written in terms of the deviatoric stresses by substituting σ_{ij}^{dev} for σ_{ij} in (14). The resulting formula gives the strain energy density caused by deviatoric (or distortional) strain alone,

$$w_e^{dev} = \frac{1}{2}\sigma_{ij}^{dev}\left(\frac{\sigma_{ij}^{dev}}{2\mu} - \frac{\lambda\delta_{ij}\sigma_{kk}^{dev}}{2\mu(2\mu+3\lambda)}\right). \tag{15}$$

Notice the second term in the parenthesis of (15). It contains the trace of the deviatoric stress, σ_{kk}^{dev}, which is zero. This can be seen by taking the trace of (8). Hence, (15) is simplified to

$$w_e^{dev} = \frac{1}{2}\sigma_{ij}^{dev}\frac{\sigma_{ij}^{dev}}{2\mu} = \frac{1}{2\mu}\left(\frac{1}{2}\sigma_{ij}^{dev}\sigma_{ij}^{dev}\right) = \frac{1}{2\mu}J_2. \tag{16}$$

Notice that $J_2 = \frac{1}{2}\sigma_{ij}^{dev}\sigma_{ij}^{dev}$. J_2 is the second invariant of the deviatoric stress tensor, σ_{ij}^{dev}, in equation (8). In a later section it is shown that the von Mises yield criterion is a function of J_2. It is for this reason that the von Mises yield criterion is sometimes referred to as J_2 plasticity.

By substituting (8) into (16) an expression is obtained in terms of stress tensor components, σ_{ij}, that is

$$w_e^{dev} = \frac{1}{2}\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right)\left(\frac{\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}}{2\mu}\right). \tag{17}$$

After some algebra (17) is simplified to

$$w_e^{dev} = \frac{1}{4\mu}\left(\sigma_{ij}\sigma_{ij} - \frac{1}{3}(\sigma_{kk})^2\right). \tag{18}$$

It is important to note that (18) allows calculation of strain energy density due to deviatoric (or distortional) strains alone. The von Mises theory of yielding hypothesizes that yielding begins when some maximum distortional strain energy density is reached. The maximum distortional strain energy density that can be reached at the onset of yielding is found in the following section.

7 Maximum Distortional Strain Energy Due to Uniaxial State of Stress

If a material point in a body is placed in uniaxial tension, like that shown in Figure 1, the *only* nonzero stress component is $\sigma_{11} = \sigma_x$. For this condition the maximum value is $\sigma_x = \sigma_{yield}$. This is the maximum stress that can be reached before the onset of yielding and at that stress state the associated distortional strains are at their maximum [1, 2]. Hence, this is a simple thought experiment that illustrates a stress state that will cause the maximum distortional strain energy density possible. The maximum distortional strain energy density possible can immediately be calculated by substituting $\sigma_{11} = \sigma_{yield}$ into (18) and noting that all other stresses are zero during the uniaxial state of stress. The result is

$$w_e^{dev} = \frac{1}{4\mu}\left((\sigma_{yield})^2 - \frac{1}{3}(\sigma_{yield})^2\right) = \frac{1}{4\mu}\left(\frac{2}{3}(\sigma_{yield})^2\right). \tag{19}$$

8 von Mises Yield Criterion

The von Mises yield criterion hypothesizes that yielding will occur for a general 3D state of stress when the combination of stresses reaches the maximum distortional strain energy density [1, 2]. Therefore, setting (18) equal to (19) yields

$$\frac{1}{4\mu}\left(\sigma_{ij}\sigma_{ij} - \frac{1}{3}(\sigma_{kk})^2\right) = \frac{1}{4\mu}\left(\frac{2}{3}(\sigma_{yield})^2\right). \tag{20}$$

Solving (20) for σ_{yield} gives

$$\sigma_{yield} = \sqrt{\frac{3}{2}\left(\sigma_{ij}\sigma_{ij} - \frac{1}{3}(\sigma_{kk})^2\right)}. \tag{21}$$

Equivalently, in terms of, σ_{ij}^{dev},

$$\sigma_{yield} = \sqrt{\frac{3}{2}\left(\sigma_{ij}^{dev}\sigma_{ij}^{dev}\right)} = \sqrt{3J_2}. \tag{22}$$

It is important to note that if the right hand side of equation (21) or (22) is less than yield than yielding has not taken place yet. It is for this reason that a von Mises stress, using (21) or (22), is often defined as

$$\sigma_v = \sqrt{\frac{3}{2}\left(\sigma_{ij}\sigma_{ij} - \frac{1}{3}(\sigma_{kk})^2\right)} = \sqrt{3J_2}. \tag{23}$$

Then, if $\sigma_v < \sigma_{yield}$ the material point in the body has not yielded. However, if $\sigma_v \geq \sigma_{yield}$ then the material point has yielded. This establishes a yield criterion to determine if yielding has occurred due to a general 3D state of stress.

Last, the von Mises stress formulas are provided in some useful forms suitable for calculations. In order to illustrate how these formulas are obtained, observe that

$$\begin{aligned}\sigma_{ij}\sigma_{ij} &= \sigma_{11}\sigma_{11} + \sigma_{12}\sigma_{12} + \sigma_{13}\sigma_{13} + \sigma_{21}\sigma_{21} + \sigma_{22}\sigma_{22} + \sigma_{23}\sigma_{23} + \sigma_{31}\sigma_{31} + \sigma_{32}\sigma_{32} + \sigma_{33}\sigma_{33}\\ &= \sigma_{11}\sigma_{11} + \sigma_{22}\sigma_{22} + \sigma_{33}\sigma_{33} + 2\sigma_{12}\sigma_{12} + 2\sigma_{13}\sigma_{13} + 2\sigma_{23}\sigma_{23}\end{aligned} \tag{24}$$

and

$$\frac{1}{3}(\sigma_{kk})^2 = \frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})^2 = \frac{1}{3}((\sigma_{11})^2+(\sigma_{22})^2+(\sigma_{33})^2+2\sigma_{11}\sigma_{22}+2\sigma_{11}\sigma_{33}+2\sigma_{22}\sigma_{33}) \tag{25}$$

If (24) and (25) are inserted into (21), after some algebra one obtains

$$\sigma_v = \sqrt{\frac{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2)}{2}}. \tag{26}$$

Equation (26) is a suitable form for computer calculations to test for yielding at various points in a body. Modern finite element analysis software often provides options for calculating and visualizing von Mises stresses as a function of position throughout a body being analyzed. Another useful form, in terms of principal stresses, is

$$\sigma_v = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}}. \tag{27}$$

9 Conclusion

The von Mises yield criterion based on maximum distortional strain energy density is derived. It provides a test by which a 3D state of stress can be tested to determine if yielding at a material point has occurred. This has important design ramifications because engineering designs often need to meet required safety margins to avoid failure by yielding. The von Mises theory of failure is particularly useful for ductile metal materials and experiments have shown that it is a good predictor of failure for such materials [2, 8]. Suitable references for further study are provided [4, 5].

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