

Nonlinear Static - 1D Plasticity - Various Forms of Isotropic Hardening
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Draft Date January 25, 2012

key words: plasticity, algorithms, nonlinear analysis, isotropic hardening, algorithmic tangent modulus, elasto-plastic tangent modulus, Kuhn-Tucker conditions, consistency parameter, internal hardening variable, equivalent plastic strain, flow rule, consistency condition, yield condition

1 Introduction

Metals loaded beyond the elastic limit deform plastically. It is often the goal in structural mechanics to model plastic deformations of such metals in a computational setting. Creating algorithms to model the combined effect of elastic and plastic deformations is not trivial. However, plasticity for the one dimensional case is a good introduction to the concepts necessary to construct such algorithms. Hence, this article presents plasticity from a one dimensional point of view and derives algorithms for a variety of different hardening models. It is the goal of this article to introduce students to algorithmic plasticity (nonlinear material) models and also pave the way for future learning in the area of three dimensional plasticity.

One dimensional plasticity is helpful as an introduction and as an intermediate step toward more difficult topics, but it is also practical for bar elements often modeled in structural analysis software. It is therefore worth noting that the information learned herein is useful for the implementation of plasticity by modifying a standard linear truss analysis program. Only small strains are considered.

2 Elastic and plastic behavior

Material testing is often accomplished by tensile tests. Plots of stress versus strain are often created from tension tests. For many materials the plot of stress versus strain is linear for stresses below a certain threshold called the yield stress. If the material is loaded beyond this point the material yields and begins to deform plastically. How the material deforms after passing the yield point varies from material to material. For instance many materials exhibit hardening after reaching the yield limit. This hardening is exhibited by increasing resistance to load after yielding, but this increase in resistance is much less than the resistance provided by the material before yielding.

If hardening is present the post yield behavior exhibits *both* increasing elastic strain and increasing plastic strain. A variety of models are available to try and simulate how the plastic strain increases with increasing stress. The post yield behavior is often modeled by specifying how the yield stress evolves with increasing *plastic* strain. A variety of such models are shown in Figure 1. In the plots the yield point is shown, $\sigma_y = 30$ ksi, and the hardening behavior is plotted based on the formulas shown. It is important to note that the plots are of stress versus plastic strain. When plots of stress versus *total* strain are created

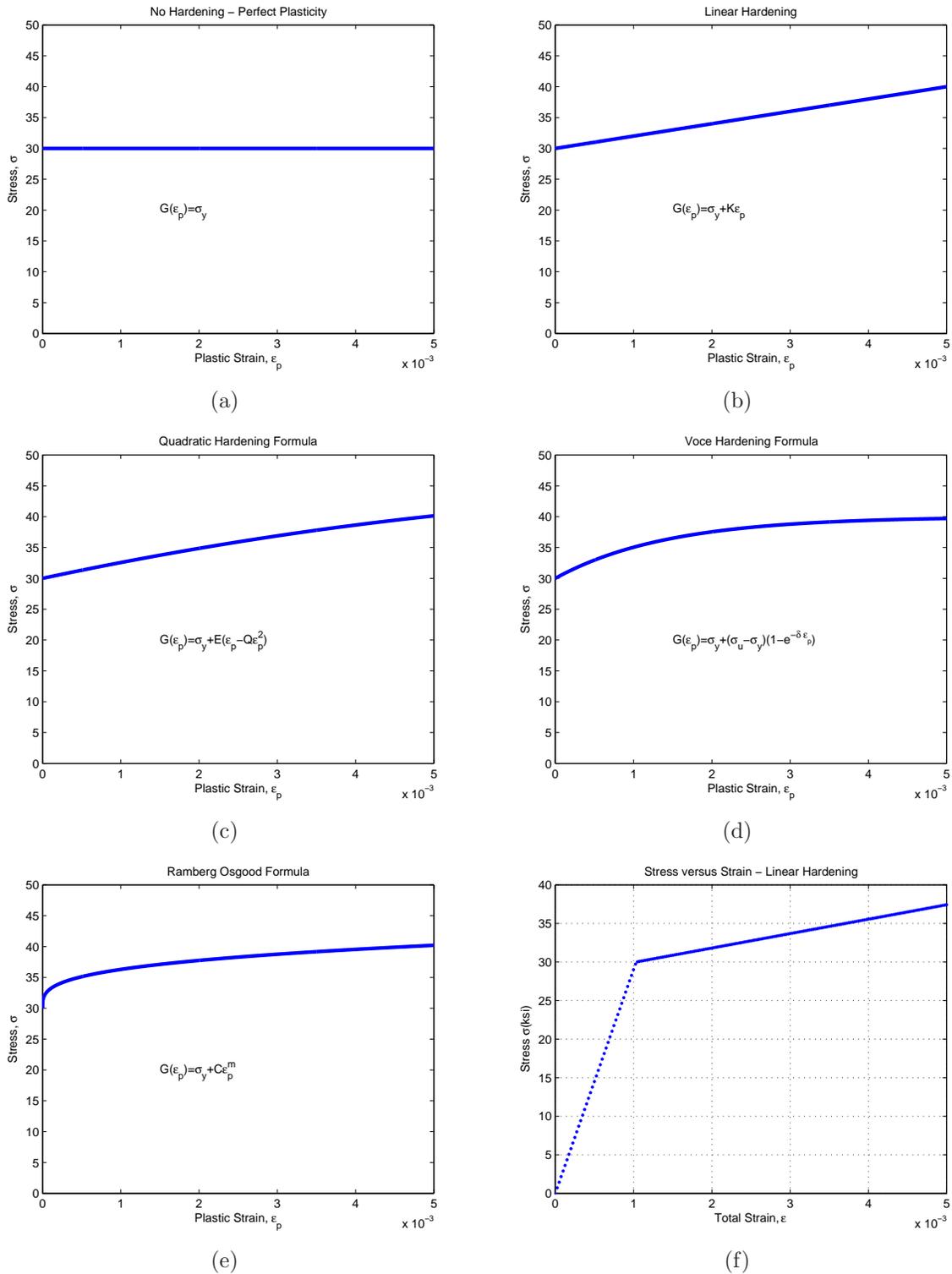


Figure 1: Plastic Strain Hardening Models ($\sigma_y = 30$ ksi): (a) No Hardening - Perfect Plasticity, (b) Linear Hardening, (c) Quadratic Hardening, (d) Voce Hardening, (e) Ramberg Osgood Hardening, (f) Stress versus total strain with linear hardening.

based on these models, the post yield behavior will look similar but will be different since the post yield behavior also accounts for increasing elastic strains. Hence, for example, in the case of Figure 1(b) the post yield slope is called the plastic modulus. Whereas a plot of stress versus total strain using linear hardening, Figure 1(f), has a post yield slope that is called the elasto-plastic modulus and it is *not* the same as the plastic modulus. Because of hardening the yield point has increased and as a result more elastic strain exists due to the higher yield point. Hence, for hardening materials, increase in plastic strain is accompanied with increase in elastic strain. In later sections the derivation of the elasto-plastic modulus is illustrated for the models given.

3 Definitions

Basic terminology helpful for understanding the concepts involved in modeling plasticity is presented in this section. The definitions given are not exhaustive, but provide at least the basic ideas. Some of the terms are easy to define clearly. Other terms can be defined but are better understood after working with the material in following sections. The notation used closely follows that given by Simo and Hughes [6].

Brief definitions of the most important terminology are as follows. Elastic strain, ε^e , is any strain that takes place before exceeding the yield stress. However, it is important to note that, for hardening materials, elastic strain is increasing during post yield also. Hence, strain that is removed during unloading is a better definition of elastic strain. Plastic strain, ε^p , is permanent strain that remains after unloading. Total strain is the sum of elastic strain and plastic strain, $\varepsilon = \varepsilon^e + \varepsilon^p$. Any time plastic strains are taking place it follows that plastic flow is taking place. When hardening occurs the value of the yield stress changes. For this reason, plasticity models use internal hardening variables as a means for keeping track of accumulated hardening. For isotropic hardening, hardening accumulates if plastic strain is positive or negative, hence the internal hardening variable keeps track of the total change in plastic strain. Therefore, the isotropic internal hardening variable α is called the equivalent plastic strain. A yield condition (a function) is used to mathematically identify when yielding happens. Often in the literature the yield condition is denoted by f . It typically includes the current level of stress minus the initial yield stress added to a function of α which describes the type of hardening. Examples of yield conditions are provided later in this paper. As plastic strain takes place, α is updated and changes, the yield condition f is calculated and is either a negative value or a positive value. If it is negative the current level of stress is below yield and only elastic strains are increasing. If it is positive the current level of stress is above the yield stress and (with the presence of hardening) elastic and plastic strains are increasing. However, in actuality $f > 0$ is not allowed and requires calculation of the amount of plastic flow and how the hardening level evolves such that $f = 0$ is achieved. This is accomplished by solving for a consistency parameter γ , which allows a means for determining the level of plastic flow and hardening such that the condition $f = 0$ is satisfied. The variable γ is called the consistency parameter because it is determined so that all relevant variables are consistent with the requirement that $f = 0$ when plastic flow is taking place. This is a rather involved process and is difficult to describe in words. The whole process of elastic loading and unloading or elasto-plastic loading and unloading requires a

careful mathematical description. This has led researchers to use Kuhn-Tucker conditions. The Kuhn-Tucker conditions are $\gamma \geq 0$, $f(\sigma) \leq 0$, and $\gamma f(\sigma) = 0$. These conditions or rules are used to construct the mathematical algorithms which model the process of plastic flow (for a more in depth discussion the reader is referred to Simo and Hughes [6]). A consistency condition is also used and is written as $\gamma \dot{f} = 0$, if $f = 0$. The sign function is often used in the mathematics of plasticity algorithms. The function takes any variable and returns -1 if the variable is negative and $+1$ if the variable is positive. Plasticity algorithms are often used in finite element analysis programs to model nonlinear material behavior. In such cases the finite element analysis program is incremental in nature. Each new load increment increases stress levels, strain levels and displacements. Hence, during each of these increments the plasticity algorithm is used at the element level in the analysis program. At the beginning of each increment (or load step) it is not known whether the new level of stress causes yielding (in a particular element) or not. For this reason it is initially assumed that the incremental stress is elastic. In the plasticity algorithms, the resulting new stress level is termed a trial stress. This trial stress is plugged into the yield condition f and the rules of the algorithm are followed as necessary. In the course of the algorithm the correct new stress level is determined (ie, the trial stress is corrected if the stress increment was not entirely elastic). In some derivations given in this paper derivatives with respect to time such as $\frac{d\alpha}{dt} = \dot{\alpha}$ are used. However, the type of material models described in this paper are rate independent, so it may seem odd to take derivatives with respect to time. Hence, in this setting the interpretation of derivatives with respect to time really mean changes with respect to a load step or load increment in a finite element analysis program. For example, if the current value of the internal variable α is α_n then during the load increment the variable may increase by an amount $\Delta\alpha$ such that the new value of α is $\alpha_{n+1} = \alpha_n + \Delta\alpha$. In this example $\Delta\alpha$ is analogous to $\dot{\alpha}$. Another way to look at this is to say that (in a rate independent material model) a load increment is like a time increment. This is the reason derivatives with respect to time are used in the literature even in rate independent material models. In some theoretical derivations γ is used to denote the consistency parameter. However, in the plasticity model algorithms $\Delta\gamma$ is used to denote the consistency parameter. Otherwise, there is no difference between γ and $\Delta\gamma$ used in this paper. For the 1D plasticity case the algorithmic tangent modulus is equivalent to the elasto-plastic modulus. In higher dimensions this is not true. For some plasticity models the yield function is complicated enough that algebraically solving for the consistency parameter is not possible and hence it must be solved for using Newton-Raphson iterations. In this process, in order to preserve the second order rate of convergence of the Newton-Raphson method (which is likely also taking place at the global level of the nonlinear finite element program) it is necessary to have a consistent variation (or derivative) of stress with respect to the total strain. To get such a consistent variation it is necessary to take the derivative of the algorithmic stress with respect to the total strain variable. The algorithmic tangent modulus is specifically derived by taking the derivative $\frac{\partial \sigma_{n+1}^{(k)}}{\partial \varepsilon_{n+1}^{(k)}} = \mathbf{C}_{n+1}^{(k)}$. This is a derivative of the algorithmic stress and hence results in the algorithmic tangent modulus, $\mathbf{C}_{n+1}^{(k)}$. How plastic strain evolves in plasticity algorithms is called a flow rule. In this paper the flow rule takes the form $\dot{\varepsilon}^p = \gamma \frac{df}{d\sigma} = \gamma \text{sign}(\sigma)$. The isotropic hardening law takes the form $\dot{\alpha} = \Delta\alpha = \gamma$.

4 1D Plasticity General Isotropic Hardening

The algorithmic pieces of one dimensional plasticity with a general expression for isotropic hardening is presented next. The derivations and notation closely follows that given by [6].

4.1 Derivation of Elasto-Plastic Tangent Modulus

As usual the modulus of elasticity is E , the equivalent plastic strain is α and the total strain is defined as

$$\varepsilon = \varepsilon^p + \varepsilon^e. \quad (4.1)$$

Stress is linear elastic when $f < 0$ and is calculated as

$$\sigma = E\varepsilon^e = E(\varepsilon - \varepsilon^p) \quad (4.2)$$

Kinematic hardening is neglected and the flow rule is assumed as

$$\dot{\varepsilon}^p = \gamma \text{sign}(\sigma). \quad (4.3)$$

The yield condition is defined as follows:

$$f(\sigma) = |\sigma| - G(\alpha) \quad (4.4)$$

,where $G(\alpha)$ is a yield stress function which includes the type of isotropic hardening, which is possibly a function of α . The customary Kuhn-Tucker conditions apply ($\gamma \geq 0$, $f(\sigma) \leq 0$, and $\gamma f(\sigma) = 0$). If $f(\sigma)$ is to be zero the consistency condition requires that $\dot{f} = 0$. Hence, when $\gamma > 0$, $\dot{f} = 0$ so that by the chain rule

$$\dot{f} = \frac{\partial f}{\partial \sigma} \frac{\partial \sigma}{\partial t} + \frac{\partial f}{\partial G} \frac{\partial G}{\partial \alpha} \frac{\partial \alpha}{\partial t} = 0. \quad (4.5)$$

Inserting each partial derivative into the above formula gives

$$\dot{f} = \text{sign}(\sigma)\dot{\sigma} + (-1)\frac{\partial G}{\partial \alpha}\dot{\alpha} = 0. \quad (4.6)$$

Recall that $\dot{\alpha} = \gamma$, $\dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^p)$ and $\dot{\varepsilon}^p = \gamma \text{sign}(\sigma)$. Substituting this information into (4.6) yields

$$\dot{f} = \text{sign}(\sigma)E(\dot{\varepsilon} - \gamma \text{sign}(\sigma)) - \frac{\partial G}{\partial \alpha}\gamma = 0. \quad (4.7)$$

Recognizing that $(\text{sign}(\sigma))^2 = 1$ and solving for γ gives

$$\gamma = \frac{\text{sign}(\sigma)E\dot{\varepsilon}}{E + \frac{\partial G}{\partial \alpha}}. \quad (4.8)$$

With the above results in hand the elasto-plastic tangent modulus, $C_{ep} = \frac{d\sigma}{d\varepsilon}$, is found as follows. Observe that $\dot{\sigma} = \frac{d\sigma}{d\varepsilon}\dot{\varepsilon}$. Then using (4.2), (4.3) and (4.8) yields

$$\dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^p) = E(\dot{\varepsilon} - \gamma \text{sign}(\sigma)) = E\left(\dot{\varepsilon} - \frac{(\text{sign}(\sigma))^2 E \dot{\varepsilon}}{E + \frac{\partial G}{\partial \alpha}}\right). \quad (4.9)$$

Upon simplification the stress rate becomes

$$\dot{\sigma} = \left[E - \frac{E^2}{E + \frac{\partial G}{\partial \alpha}} \right] \dot{\varepsilon} = \frac{E \frac{\partial G}{\partial \alpha}}{E + \frac{\partial G}{\partial \alpha}} \dot{\varepsilon}. \quad (4.10)$$

By inspection of (4.10) the elasto-plastic tangent modulus is

$$\mathbf{C}_{ep} = \frac{E \frac{\partial G}{\partial \alpha}}{E + \frac{\partial G}{\partial \alpha}}. \quad (4.11)$$

4.2 Development of algorithmic ingredients for 1D plasticity with general isotropic hardening

Consider now the algorithmic ingredients for a 1D plasticity problem. Suppose that a strain increment is provided so that a new total strain, ε , is given. From this information a trial value of the yield condition, f_{n+1}^{trial} , is calculated. If $f_{n+1}^{trial} \leq 0$ the strain is elastic and the solution is trivial. If on the other hand $f_{n+1}^{trial} > 0$ then a plastic step has occurred. For a plastic step, the problem is to find σ_{n+1} , α_{n+1} such that $f(\sigma_{n+1}, \alpha_{n+1}) = 0$ and $\Delta\gamma > 0$. Having this information then allows calculation of ε_{n+1}^p , which is the new total plastic strain. To derive an algorithm for a typical plastic step first express σ_{n+1} as a function of σ_{n+1}^{trial} and $\Delta\gamma$ as follows:

$$\begin{aligned} \sigma_{n+1} &= E(\varepsilon_{n+1} - \varepsilon_{n+1}^p) \\ &= E(\varepsilon_{n+1} - \varepsilon_n^p) - E(\varepsilon_{n+1}^p - \varepsilon_n^p) \\ &= E(\varepsilon_{n+1} - \varepsilon_n^p) - E\dot{\varepsilon}_{n+1}^p \\ &= \sigma_{n+1}^{trial} - E\Delta\gamma sign(\sigma_{n+1}). \end{aligned} \quad (4.12)$$

Assuming that the correct value of $\Delta\gamma > 0$ can be found for the current plastic step, all the quantities are calculated as

$$\sigma_{n+1} = \sigma_{n+1}^{trial} - \Delta\gamma E sign(\sigma_{n+1}) \quad (4.13a)$$

$$\varepsilon_{n+1}^p = \varepsilon_n^p + \Delta\gamma sign(\sigma_{n+1}) \quad (4.13b)$$

$$\alpha_{n+1} = \alpha_n + \Delta\gamma \quad (4.13c)$$

$$f_{n+1} \equiv |\sigma_{n+1}| - G(\alpha_{n+1}) = 0. \quad (4.13d)$$

The set of equations (4.13) are solved in terms of the trial elastic state (see Figure 2) as follows:

$$|\sigma_{n+1}| sign(\sigma_{n+1}) = |\sigma_{n+1}^{trial}| sign(\sigma_{n+1}^{trial}) - \Delta\gamma E sign(\sigma_{n+1}) \quad (4.14)$$

,which is rearranged to give

$$[|\sigma_{n+1}| + \Delta\gamma E] sign(\sigma_{n+1}) = |\sigma_{n+1}^{trial}| sign(\sigma_{n+1}^{trial}). \quad (4.15)$$

Now, since $\Delta\gamma$ and E are greater than zero, for the above equation to be valid the following two conditions must be true

$$sign(\sigma_{n+1}) = sign(\sigma_{n+1}^{trial}) \quad (4.16a)$$

$$|\sigma_{n+1}| + \Delta\gamma E = |\sigma_{n+1}^{trial}|. \quad (4.16b)$$

4.3 The Algorithmic Tangent Modulus

Before summarizing the algorithm, the *consistent* or *algorithmic tangent modulus* is derived. The algorithmic tangent modulus is

$$\mathbf{C}_{n+1}^{(k)} = \frac{\partial \sigma_{n+1}^{(k)}}{\partial \varepsilon_{n+1}^{(k)}}. \quad (4.20)$$

In the following derivation the superscript k is omitted. Some preliminary results are obtained for the derivation. First, differentiate the trial stress σ_{n+1}^{trial} to get

$$\frac{\partial \sigma_{n+1}^{trial}}{\partial \varepsilon_{n+1}} = E. \quad (4.21)$$

Next, obtain $\frac{\partial(\Delta\gamma)}{\partial \varepsilon_{n+1}}$ by realizing that (4.8) implies the following:

$$\frac{\dot{\gamma}}{\dot{\varepsilon}} = \frac{sign(\sigma)E}{E + \frac{\partial G}{\partial \alpha}} \Rightarrow \frac{\partial(\Delta\gamma)}{\partial \varepsilon_{n+1}} = \frac{sign(\sigma_{n+1}^{trial})E}{E + \frac{\partial G}{\partial \alpha_{n+1}}}. \quad (4.22)$$

The above observation that leads to (4.22) is not rigorous and is not the standard way to obtain it (however, it is easier than the rigorous approach, below, and seems to also work in higher dimensions). A more rigorous approach described in [6] is to start with the scalar consistency condition. For example, use the first line of (4.17) set equal to zero, rearrange the equation, use implicit differentiation with the chain rule as needed and solve for $\frac{\partial(\Delta\gamma)}{\partial \varepsilon_{n+1}}$ as follows:

$$|\sigma_{n+1}^{trial}| - \Delta\gamma E - G(\alpha_{n+1}) = 0 \quad (4.23)$$

$$\frac{\partial(\Delta\gamma)}{\partial \varepsilon_{n+1}} E + \frac{\partial G}{\partial \alpha_{n+1}} \frac{\partial \alpha_{n+1}}{\partial(\Delta\gamma)} \frac{\partial(\Delta\gamma)}{\partial \varepsilon_{n+1}} = \frac{\partial |\sigma_{n+1}^{trial}|}{\partial \varepsilon_{n+1}} \quad (4.24)$$

$$\frac{\partial(\Delta\gamma)}{\partial \varepsilon_{n+1}} E + \frac{\partial G}{\partial \alpha_{n+1}} (1) \frac{\partial(\Delta\gamma)}{\partial \varepsilon_{n+1}} = sign(\sigma_{n+1}^{trial}) E \quad (4.25)$$

$$\frac{\partial(\Delta\gamma)}{\partial \varepsilon_{n+1}} \left(E + \frac{\partial G}{\partial \alpha_{n+1}} \right) = sign(\sigma_{n+1}^{trial}) E \quad (4.26)$$

$$\frac{\partial(\Delta\gamma)}{\partial \varepsilon_{n+1}} = \frac{sign(\sigma_{n+1}^{trial}) E}{E + \frac{\partial G}{\partial \alpha_{n+1}}} \quad (4.27)$$

Now, with the above results, differentiate (4.18)a with respect to ε_{n+1} to get

$$\frac{\partial \sigma_{n+1}}{\partial \varepsilon_{n+1}} = \left[1 - \frac{\Delta\gamma E}{|\sigma_{n+1}^{trial}|} \right] \frac{\partial \sigma_{n+1}^{trial}}{\partial \varepsilon_{n+1}} + \frac{\partial \left(\frac{-\Delta\gamma E}{|\sigma_{n+1}^{trial}|} \right)}{\partial \varepsilon_{n+1}} \sigma_{n+1}^{trial}. \quad (4.28)$$

To simplify (4.28), by using (4.21) and (4.22), find

$$\begin{aligned} \frac{\partial \left(\frac{-\Delta\gamma E}{|\sigma_{n+1}^{trial}|} \right)}{\partial \varepsilon_{n+1}} &= \frac{\partial(-\Delta\gamma)}{\partial \varepsilon_{n+1}} \frac{E}{|\sigma_{n+1}^{trial}|} + (-\Delta\gamma E) \left(\frac{\partial |\sigma_{n+1}^{trial}|^{-1}}{\partial \varepsilon_{n+1}} \right) \\ &= \frac{-E^2}{E + \frac{\partial G}{\partial \alpha_{n+1}}} \frac{sign(\sigma_{n+1}^{trial})}{|\sigma_{n+1}^{trial}|} + (-\Delta\gamma E) \left(\frac{-1}{|\sigma_{n+1}^{trial}|^2} \right) sign(\sigma_{n+1}^{trial}) E. \end{aligned} \quad (4.29)$$

Substitute the result of (4.29) into (4.28) and use (4.22) to obtain

$$\begin{aligned}
\frac{\partial \sigma_{n+1}}{\partial \varepsilon_{n+1}} &= \left[1 - \frac{\Delta \gamma E}{|\sigma_{n+1}^{trial}|} \right] E - \frac{E^2 \text{sign}(\sigma_{n+1}^{trial}) \sigma_{n+1}^{trial}}{\left(E + \frac{\partial G}{\partial \alpha_{n+1}}\right) |\sigma_{n+1}^{trial}|} + \frac{\Delta \gamma E^2 \text{sign}(\sigma_{n+1}^{trial}) \sigma_{n+1}^{trial}}{|\sigma_{n+1}^{trial}|^2} \\
&= E - \frac{\Delta \gamma E^2}{|\sigma_{n+1}^{trial}|} - \frac{E^2 |\sigma_{n+1}^{trial}|}{\left(E + \frac{\partial G}{\partial \alpha_{n+1}}\right) |\sigma_{n+1}^{trial}|} + \frac{\Delta \gamma E^2 |\sigma_{n+1}^{trial}|}{|\sigma_{n+1}^{trial}|^2} \\
&= E - \frac{\Delta \gamma E^2}{|\sigma_{n+1}^{trial}|} - \frac{E^2}{E + \frac{\partial G}{\partial \alpha_{n+1}}} + \frac{\Delta \gamma E^2}{|\sigma_{n+1}^{trial}|} \\
&= \frac{E \frac{\partial G}{\partial \alpha_{n+1}}}{E + \frac{\partial G}{\partial \alpha_{n+1}}} \\
&= \frac{E \frac{\partial G}{\partial \alpha}}{E + \frac{\partial G}{\partial \alpha}}.
\end{aligned} \tag{4.30}$$

Therefore, in the 1D case, the algorithmic tangent modulus is equivalent to the elasto-plastic tangent modulus, that is

$$\mathbf{C}_{ep} = \mathbf{C}_{n+1}^{(k)} = \frac{\partial \sigma_{n+1}^{(k)}}{\partial \varepsilon_{n+1}^{(k)}} = \frac{E \frac{\partial G}{\partial \alpha}}{E + \frac{\partial G}{\partial \alpha}}. \tag{4.31}$$

In higher dimensions they are not equivalent.

4.4 Summary of results

With all of the above in hand it is possible to summarize the algorithm for 1D plasticity with general isotropic hardening. The algorithm is summarized in Box 4.1.

5 Examples of different types of *isotropic* hardening

Using the results of section 4.4, different types of hardening functions used to create the yield stress function $G(\alpha)$ are examined. The different cases presented below provide a broad sampling of hardening models used for materials. These cases also demonstrate the actual implementation of the plasticity algorithm of Box 4.1. The yield stress function and its derivative are substituted into the algorithm for general isotropic hardening at the appropriate locations. These algorithms advance the solution variables from their current values at step n to their values at step $n+1$. The resulting algorithm is summarized in a box for each case given below. After observing the variety of examples given below it is hoped that the reader can then tackle any case.

5.1 Perfect Plasticity (no hardening)

For this case there is no hardening. The yield stress function is $G(\alpha) = \sigma_y$ and therefore $\frac{\partial G}{\partial \alpha} = 0$. The algorithm is summarized in Box 5.1. Note that the variable α is not needed for this case of perfect plasticity. In this case, in step 4 of the algorithm, the consistency parameter, $\Delta \gamma$, is directly solved for algebraically.

1. Start with stored known variables $\{\varepsilon_n, \varepsilon_n^p, \alpha_n\}$.
2. An increment of strain gives $\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n$.
3. Calculate the elastic trial stress, the trial value for the yield function and check for yielding.

$$\sigma_{n+1}^{trial} = E(\varepsilon_{n+1} - \varepsilon_n^p)$$

$$f_{n+1}^{trial} = |\sigma_{n+1}^{trial}| - G(\alpha_n)$$

If $f_{n+1}^{trial} \leq 0$ then the load step is elastic

$$\text{set } \sigma_{n+1} = \sigma_{n+1}^{trial}$$

$$\text{set } C_{ep} = E$$

EXIT the algorithm

Else $f_{n+1}^{trial} > 0$ and hence the load step is elasto-plastic

continue at step 4

4. Elasto-plastic step

Using $f_{n+1}^{trial} - \Delta\gamma E - G(\alpha_{n+1}) + G(\alpha_n) = 0$, solve for $\Delta\gamma$.

$$\sigma_{n+1} = \left[1 - \frac{\Delta\gamma E}{|\sigma_{n+1}^{trial}|}\right] \sigma_{n+1}^{trial}$$

$$\varepsilon_{n+1}^p = \varepsilon_n^p + \Delta\gamma \text{sign}(\sigma_{n+1}^{trial})$$

$$\alpha_{n+1} = \alpha_n + \Delta\gamma$$

$$C_{ep} = \frac{E \frac{\partial G}{\partial \alpha}}{E + \frac{\partial G}{\partial \alpha}}$$

EXIT the algorithm

Box 4.1: 1D Plasticity Algorithm With General Isotropic Hardening

1. Start with stored known variables $\{\varepsilon_n, \varepsilon_n^p\}$.
2. An increment of strain gives $\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n$.
3. Calculate the elastic trial stress, the trial value for the yield function and check for yielding.

$$\sigma_{n+1}^{trial} = E(\varepsilon_{n+1} - \varepsilon_n^p)$$

$$f_{n+1}^{trial} = |\sigma_{n+1}^{trial}| - \sigma_y$$

If $f_{n+1}^{trial} \leq 0$ then the load step is elastic

$$\text{set } \sigma_{n+1} = \sigma_{n+1}^{trial}$$

$$\text{set } C_{ep} = E$$

EXIT the algorithm

Else $f_{n+1}^{trial} > 0$ and hence the load step is elasto-plastic

continue at step 4

4. Elasto-plastic step

Using $f_{n+1}^{trial} - \Delta\gamma E = 0$, solve for $\Delta\gamma$.

$$\Rightarrow \Delta\gamma = \frac{f_{n+1}^{trial}}{E}$$

$$\sigma_{n+1} = \left[1 - \frac{\Delta\gamma E}{|\sigma_{n+1}^{trial}|}\right] \sigma_{n+1}^{trial}$$

$$\varepsilon_{n+1}^p = \varepsilon_n^p + \Delta\gamma \text{sign}(\sigma_{n+1}^{trial})$$

$$C_{ep} = 0$$

EXIT the algorithm

Box 5.1: 1D Plasticity Algorithm With No Hardening (Perfect Plasticity)

5.2 Linear Hardening

Linear hardening is a fairly simple and common form. The yield stress function is $G(\alpha) = \sigma_y + K\alpha$ and therefore $\frac{\partial G}{\partial \alpha} = K$. The algorithm is summarized in Box 5.2. The consistency parameter, $\Delta\gamma$, is solved for using the last line of (4.17) as follows:

$$\begin{aligned}
 & f_{n+1}^{trial} - \Delta\gamma E - (\sigma_y + K\alpha_{n+1}) + (\sigma_y + K\alpha_n) \\
 &= f_{n+1}^{trial} - \Delta\gamma E - \sigma_y - K\alpha_{n+1} + \sigma_y + K\alpha_n \\
 &= f_{n+1}^{trial} - \Delta\gamma E - K(\alpha_{n+1} - \alpha_n) \\
 &= f_{n+1}^{trial} - \Delta\gamma E - K(\Delta\gamma) = 0.
 \end{aligned} \tag{5.1}$$

From the last line of equation (5.1)

$$\Delta\gamma = \frac{f_{n+1}^{trial}}{E + K}, \tag{5.2}$$

which is indicated in step 4 of the algorithm given in Box 5.2.

5.3 Quadratic Hardening

The following case does not seem to be common or practical. However, it is an interesting case as an academic exercise. The yield stress function is $G(\alpha) = \sigma_y + E(\alpha - Q\alpha^2)$ and therefore $\frac{\partial G}{\partial \alpha} = E(1 - 2Q\alpha)$. The algorithm is summarized in Box 5.3. The consistency parameter, $\Delta\gamma$, is solved for using the last line of (4.17) as follows:

$$f_{n+1}^{trial} - \Delta\gamma E - G(\alpha_{n+1}) + G(\alpha_n) = 0. \tag{5.3}$$

Substitution of G into (5.3), some lengthy algebra and use of the relation $\Delta\gamma = \alpha_{n+1} - \alpha_n$ results in

$$f_{n+1}^{trial} - \Delta\gamma E - E\Delta\gamma + 2\alpha_n EQ\Delta\gamma + EQ(\Delta\gamma)^2 = 0. \tag{5.4}$$

Finally, rearranging and collecting like terms gives

$$\begin{aligned}
 & a(\Delta\gamma)^2 + b\Delta\gamma + c = 0 \\
 & a = QE \\
 & b = 2\alpha_n QE - 2E \\
 & c = f_{n+1}^{trial}.
 \end{aligned} \tag{5.5}$$

Equation (5.5) is solved by using the quadratic equation

$$\Delta\gamma = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{5.6}$$

from which the positive root is chosen as the value for $\Delta\gamma$.

5.4 Exponential Hardening

Voce [7] proposed an exponential form of hardening. This assumes that the hardening eventually reaches a specified saturation (or maximum) stress. In this case the yield stress function becomes $G(\alpha) = \sigma_y + (\sigma_u - \sigma_y)(1 - e^{-\delta\alpha})$ and therefore $\frac{\partial G}{\partial \alpha} = (\sigma_u - \sigma_y)\delta e^{-\delta\alpha}$.

1. Start with stored known variables $\{\varepsilon_n, \varepsilon_n^p, \alpha_n\}$.
2. An increment of strain gives $\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n$.
3. Calculate the elastic trial stress, the trial value for the yield function and check for yielding.

$$\sigma_{n+1}^{trial} = E(\varepsilon_{n+1} - \varepsilon_n^p)$$

$$\varepsilon_{n+1}^{p\ trial} = \varepsilon_n^p$$

$$\alpha_{n+1}^{trial} = \alpha_n$$

$$f_{n+1}^{trial} = |\sigma_{n+1}^{trial}| - (\sigma_y + K\alpha_n)$$

If $f_{n+1}^{trial} \leq 0$ then the load step is elastic

$$\text{set } \sigma_{n+1} = \sigma_{n+1}^{trial}$$

$$\text{set } C_{ep} = E$$

EXIT the algorithm

Else $f_{n+1}^{trial} > 0$ and hence the load step is elasto-plastic

continue at step 4

4. Elasto-plastic step

$$\Delta\gamma = \frac{f_{n+1}^{trial}}{E+K}$$

$$\sigma_{n+1} = \left[1 - \frac{\Delta\gamma E}{|\sigma_{n+1}^{trial}|}\right] \sigma_{n+1}^{trial}$$

$$\varepsilon_{n+1}^p = \varepsilon_n^p + \Delta\gamma \text{sign}(\sigma_{n+1}^{trial})$$

$$\alpha_{n+1} = \alpha_n + \Delta\gamma$$

$$C_{ep} = \frac{EK}{E+k}$$

EXIT the algorithm

Box 5.2: 1D Plasticity Algorithm With Linear Isotropic Hardening [6]

1. Start with stored known variables $\{\varepsilon_n, \varepsilon_n^p, \alpha_n\}$.
2. An increment of strain gives $\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n$.
3. Calculate the elastic trial stress, the trial value for the yield function and check for yielding.

$$\sigma_{n+1}^{trial} = E(\varepsilon_{n+1} - \varepsilon_n^p)$$

$$\varepsilon_{n+1}^{p\ trial} = \varepsilon_n^p$$

$$\alpha_{n+1}^{trial} = \alpha_n$$

$$f_{n+1}^{trial} = |\sigma_{n+1}^{trial}| - (\sigma_y + E(\alpha_n - Q\alpha_n^2))$$

If $f_{n+1}^{trial} \leq 0$ then the load step is elastic

$$\text{set } \sigma_{n+1} = \sigma_{n+1}^{trial}$$

$$\text{set } C_{ep} = E$$

EXIT the algorithm

Else $f_{n+1}^{trial} > 0$ and hence the load step is elasto-plastic

continue at step 4

4. Elasto-plastic step

Use the positive result of $\Delta\gamma = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$a = QE, \quad b = 2\alpha_n QE - 2E, \quad c = f_{n+1}^{trial}$$

$$\sigma_{n+1} = \left[1 - \frac{\Delta\gamma E}{|\sigma_{n+1}^{trial}|} \right] \sigma_{n+1}^{trial}$$

$$\varepsilon_{n+1}^p = \varepsilon_n^p + \Delta\gamma \text{sign}(\sigma_{n+1}^{trial})$$

$$\alpha_{n+1} = \alpha_n + \Delta\gamma$$

$$C_{ep} = \frac{2Q\alpha_{n+1}E - E}{2Q\alpha_{n+1} - 2}$$

EXIT the algorithm

Box 5.3: 1D Plasticity Algorithm With Quadratic Isotropic Hardening

1. Start with stored known variables $\{\varepsilon_n, \varepsilon_n^p, \alpha_n\}$.
2. An increment of strain gives $\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n$.
3. Calculate the elastic trial stress, the trial value for the yield function and check for yielding.

$$\sigma_{n+1}^{trial} = E(\varepsilon_{n+1} - \varepsilon_n^p)$$

$$\varepsilon_{n+1}^{p\ trial} = \varepsilon_n^p$$

$$\alpha_{n+1}^{trial} = \alpha_n$$

$$C = \sigma_u - \sigma_y$$

$$f_{n+1}^{trial} = |\sigma_{n+1}^{trial}| - (\sigma_y + C(1 - e^{-\delta\alpha_n}))$$

If $f_{n+1}^{trial} \leq 0$ then the load step is elastic

$$\text{set } \sigma_{n+1} = \sigma_{n+1}^{trial}$$

$$\text{set } C_{ep} = E$$

EXIT the algorithm

Else $f_{n+1}^{trial} > 0$ and hence the load step is elasto-plastic

continue at step 4

4. Elasto-plastic step

Using $f_{n+1}^{trial} - \Delta\gamma E - G(\alpha_{n+1}) + G(\alpha_n) = 0$, solve for $\Delta\gamma$ by using Newton-Raphson iterations (see Box 5.5)

$$\sigma_{n+1} = \left[1 - \frac{\Delta\gamma E}{|\sigma_{n+1}^{trial}|}\right] \sigma_{n+1}^{trial}$$

$$\varepsilon_{n+1}^p = \varepsilon_n^p + \Delta\gamma \text{sign}(\sigma_{n+1}^{trial})$$

$$\alpha_{n+1} = \alpha_n + \Delta\gamma$$

$$C_{ep} = \frac{EC\delta e^{-\delta\alpha_{n+1}}}{E + C\delta e^{-\delta\alpha_{n+1}}}$$

EXIT the algorithm

Box 5.4: 1D Plasticity Algorithm With Exponential Isotropic Hardening

1. Set $\Delta\gamma = 0$
2. Calculate R , where $R = f_{n+1}^{trial} - \Delta\gamma E - G(\alpha_n + \Delta\gamma) + G(\alpha_n)$,
and $G(\alpha) = \sigma_y + C(1 - e^{-\delta\alpha})$
3. Initialize variables
 - set $maxiter = 10$
 - set $k = 0$, (the iteration counter)
 - set $tol = 10^{-5}$
 - set $d_g = 0$
4. WHILE $|R| > tol$ and $k < maxiter$
 - $\frac{dR}{d\Delta\gamma} = -E - C\delta e^{-\delta(\alpha_n + \Delta\gamma)}$
 - $d_g = - \left[\frac{dR}{d\Delta\gamma} \right]^{-1} R$
 - Update $\Delta\gamma = \Delta\gamma + d_g$
 - Recalculate $R = f_{n+1}^{trial} - \Delta\gamma E - G(\alpha_n + \Delta\gamma) + G(\alpha_n)$
 - Update $k = k + 1$
- END WHILE
5. END Newton-Raphson iterations

Box 5.5: Newton-Raphson iterations for exponential hardening algorithm

5.5 Ramberg-Osgood Hardening

A very common material hardening model for metals is the Ramberg-Osgood [5] model, the form of the equation used here is given by Kojic and Bathe [3]. The yield stress function is $G(\alpha) = \sigma_y + C\alpha^m$ and therefore $\frac{\partial G}{\partial \alpha} = mC\alpha^{m-1}$.

1. Start with stored known variables $\{\varepsilon_n, \varepsilon_n^p, \alpha_n\}$.
2. An increment of strain gives $\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n$.
3. Calculate the elastic trial stress, the trial value for the yield function and check for yielding.

$$\sigma_{n+1}^{trial} = E(\varepsilon_{n+1} - \varepsilon_n^p)$$

$$\varepsilon_{n+1}^{p\ trial} = \varepsilon_n^p$$

$$\alpha_{n+1}^{trial} = \alpha_n$$

$$f_{n+1}^{trial} = |\sigma_{n+1}^{trial}| - (\sigma_y + C\alpha_n^m)$$

If $f_{n+1}^{trial} \leq 0$ then the load step is elastic

$$\text{set } \sigma_{n+1} = \sigma_{n+1}^{trial}$$

$$\text{set } C_{ep} = E$$

EXIT the algorithm

Else $f_{n+1}^{trial} > 0$ and hence the load step is elasto-plastic

continue at step 4

4. Elasto-plastic step

Using $f_{n+1}^{trial} - \Delta\gamma E - G(\alpha_{n+1}) + G(\alpha_n) = 0$, solve for $\Delta\gamma$ by using Newton-Raphson iterations (see Box 5.7)

$$\sigma_{n+1} = \left[1 - \frac{\Delta\gamma E}{|\sigma_{n+1}^{trial}|}\right] \sigma_{n+1}^{trial}$$

$$\varepsilon_{n+1}^p = \varepsilon_n^p + \Delta\gamma \text{sign}(\sigma_{n+1}^{trial})$$

$$\alpha_{n+1} = \alpha_n + \Delta\gamma$$

$$C_{ep} = \frac{EmC\alpha_{n+1}^{m-1}}{E+mC\alpha_{n+1}^{m-1}}$$

EXIT the algorithm

Box 5.6: 1D Plasticity Algorithm With Ramberg-Osgood Isotropic Hardening

1. Set $\Delta\gamma = 0$
2. Calculate R , where $R = f_{n+1}^{trial} - \Delta\gamma E - G(\alpha_n + \Delta\gamma) + G(\alpha_n)$,
and $G(\alpha) = \sigma_y + C\alpha^m$
3. Initialize variables
 - set $maxiter = 10$
 - set $k = 0$ (the iteration counter)
 - set $tol = 10^{-5}$
 - set $d_g = 0$
4. WHILE $|R| > tol$ and $k < maxiter$
 - $\frac{dR}{d\Delta\gamma} = -E - mC(\alpha_n + \Delta\gamma)^{m-1}$
 - $d_g = - \left[\frac{dR}{d\Delta\gamma} \right]^{-1} R$
 - Update $\Delta\gamma = \Delta\gamma + d_g$
 - Recalculate $R = f_{n+1}^{trial} - \Delta\gamma E - G(\alpha_n + \Delta\gamma) + G(\alpha_n)$
 - Update $k = k + 1$
 END WHILE
5. END Newton-Raphson iterations

Box 5.7: Newton-Raphson iterations for Ramberg-Osgood hardening algorithm

6 Computer Implementation and Results

A truss program is implemented in MATLAB which includes the possibility of plastic deformations. Only small strains are considered in the implemented program. Several examples are provided. In all cases only isotropic hardening is considered.

6.1 Outline of Computer Algorithm – Displacement Control

The following implicit algorithm uses Newton-Raphson iterations within each specified displacement increment to enforce global equilibrium for the truss structure (see Clarke and Hancock [1] and McGuire et al [4] for displacement control details). The specified displacement increments are prescribed at a structure dof chosen by the user. Typically this is the structure dof of maximum displacement in the chosen dof direction. In the algorithm presented, equal size displacement increments are used. The reader is encouraged to note the specific locations where the 1D plasticity routines are introduced into the algorithm. Without the introduction of the plasticity algorithms the computer program would be linear. The algorithm proceeds as follows:

1. Define/initialize variables

- D_{max} = the user specified maximum displacement at dof q
- $ninc$ = the user specified number of displacement increments to reach D_{max}
- $\Delta\bar{u}_q = D_{max}/ninc$ = the specified incremental displacement at dof q
- \mathbf{F} = the total vector of externally applied global nodal forces
- \mathbf{F}^{n+1} = the current externally applied global nodal force vector
- λ^{n+1} = the current load ratio, that is $\lambda^{n+1}\mathbf{F} = \mathbf{F}^n + d\mathbf{F} = \mathbf{F}^n + d\lambda^{n+1}\mathbf{F} = \mathbf{F}^{n+1}$, the load ratio starts out equal to zero
- \mathbf{N} = the vector of truss axial forces, axial force in truss element i is N_i
- \mathbf{u} = the vector of global nodal displacements, initially $\mathbf{u} = \mathbf{0}$
- \mathbf{x} = the vector of nodal x coordinates in the undeformed configuration
- \mathbf{y} = the vector of nodal y coordinates in the undeformed configuration
- \mathbf{L} = the vector of truss element lengths based on the current \mathbf{u} . L for truss element i is $L_i = \sqrt{((x_2 + u_{x2}) - (x_1 + u_{x1}))^2 + ((y_2 + u_{y2}) - (y_1 + u_{y1}))^2}$. The original element lengths are saved in a vector \mathbf{L}_o , where $L_{oi} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. The subscripts 1 and 2 refer to node 1 and 2 respectively for a given truss element.
- \mathbf{c} and \mathbf{s} = the vectors of cosines and sines for each truss element angle based on the undeformed truss.
- $\mathbf{K} = \mathbf{K}_M$, the assembled global tangent stiffness matrix, where \mathbf{K}_M is the material stiffness which evolves as plastic deformations accumulate in individual truss elements in the truss.

- \mathbf{K}_s = the modified global tangent stiffness matrix to account for supports. Rows and columns associated with zero displacement dofs are set to zero and the diagonal position is set to 1. Other (more efficient) schemes are possible, but this proves simple to implement
- $\boldsymbol{\sigma}^{n+1}$ = the vector of element axial stresses
- $\boldsymbol{\varepsilon}_{n+1} = \boldsymbol{\varepsilon}_{n+1}^{elast} + \boldsymbol{\varepsilon}_{n+1}^p$ = the vector of *total* axial strain values for each element i , where $\varepsilon = \frac{1}{L_o} \frac{L^2 - L_o^2}{L + L_o}$. Note that this form of calculating strain is better conditioned for numerical calculations and is recommended by Crisfield [2].
- $\boldsymbol{\varepsilon}_{n+1}^p$ = the vector of total axial *plastic* strain values for each element i
- \mathbf{C}_{ep}^{n+1} = the vector of elasto-plastic moduli for each truss element i
- $\boldsymbol{\alpha}^{n+1}$ = the vector of equivalent plastic strain variables α_i^{n+1} for each truss element i

2. **Start Loop** over load increments (for $n = 0$ to $ninc - 1$).

- (a) Calculate global stiffness matrix \mathbf{K} based on \mathbf{L}_o , \mathbf{c} , \mathbf{s} and current value of \mathbf{C}_{ep} .
- (b) Modify \mathbf{K} to account for supports and get \mathbf{K}_s .
- (c) Calculate the incremental load ratio $d\lambda^{n+1}$. The incremental load ratio is calculated as follows. Calculate a displacement vector based on the current stiffness, that is $\hat{\mathbf{u}} = \mathbf{K}_s^{-1}\mathbf{F}$. Take from $\hat{\mathbf{u}}$ the displacement in the direction of dof q , that is \hat{u}_q . Then $d\lambda^{n+1} = \Delta\bar{u}_q/\hat{u}_q$. Update load ratio $\lambda^{n+1} = \lambda^n + d\lambda^{n+1}$.
- (d) Calculate the incremental force vector $d\mathbf{F} = d\lambda^{n+1}\mathbf{F}$.
- (e) Solve for the incremental global nodal displacements $d\mathbf{u} = \mathbf{K}_s^{-1}d\mathbf{F}$
- (f) Update global nodal displacements, $\mathbf{u}^{n+1} = \mathbf{u}^n + d\mathbf{u}$
- (g) Update $\varepsilon_i = \frac{1}{L_o} \frac{L^2 - L_o^2}{L + L_o}$ for each element i and store in $\boldsymbol{\varepsilon}_{n+1}$, note L here is based on \mathbf{u}^{n+1} .
- (h) Use chosen plasticity algorithm here to update $\boldsymbol{\varepsilon}_{n+1}^p$, $\boldsymbol{\sigma}^{n+1}$, $\boldsymbol{\alpha}^{n+1}$ and \mathbf{C}_{ep}^{n+1}
- (i) Calculate the vector of new internal truss element axial forces \mathbf{N}^{n+1} . For truss element i the axial force is $N_i^{n+1} = \sigma_i^{n+1}A_i$.
- (j) Construct the vector of internal global forces \mathbf{F}_{int}^{n+1} based on \mathbf{N}^{n+1} .
- (k) Calculate the residual $\mathbf{R} = \lambda^{n+1}\mathbf{F} - \mathbf{F}_{int}^{n+1}$ and modify the residual to account for the required supports.
- (l) Calculate the norm of the residual $R = \sqrt{\mathbf{R} \bullet \mathbf{R}}$
- (m) Iterate for equilibrium if necessary. Set up iteration variables.
 - Iteration variable = $k = 0$
 - *tolerance* = 10^{-6}
 - *maxiter* = 100
 - $\delta\mathbf{u} = \mathbf{0}$
 - $\delta\lambda = 0$

- Save $\boldsymbol{\varepsilon}_{n+1}^p$ and $\boldsymbol{\alpha}^{n+1}$ as $\boldsymbol{\varepsilon}_{on+1}^p$ and $\boldsymbol{\alpha}_o^{n+1}$
- (n) **Start Iterations** while $R > \textit{tolerance}$ and $k < \textit{maxiter}$
- i. Calculate the new global stiffness \mathbf{K} based on \mathbf{L}_o , \mathbf{c} , \mathbf{s} and current value of \mathbf{C}_{ep} .
 - ii. Modify the global stiffness to account for supports which gives \mathbf{K}_s
 - iii. Calculate the load ratio correction $\delta\lambda_{k+1}$. The load ratio correction is calculated as follows. Calculate $\check{\mathbf{u}} = \mathbf{K}_s^{-1}\mathbf{R}$ and $\hat{\mathbf{u}} = \mathbf{K}_s^{-1}\mathbf{F}$. From $\check{\mathbf{u}}$ and $\hat{\mathbf{u}}$ extract the component of displacement in the direction of dof q , that is \check{u}_q and \hat{u}_q . Then $\delta\lambda_{k+1} = \delta\lambda_k - \check{u}_q/\hat{u}_q$.
 - iv. Calculate the correction to \mathbf{u}^{n+1} , which is $\delta\mathbf{u}^{k+1} = \delta\mathbf{u}^k + \mathbf{K}_s^{-1}[\mathbf{R} - (\check{u}_q/\hat{u}_q)\mathbf{F}]$, but note that \mathbf{u}^{n+1} is not updated until all iterations are completed
 - v. Update $\varepsilon_i = \frac{1}{L_o} \frac{L^2 - L_o^2}{L + L_o}$ for each element i and store in $\boldsymbol{\varepsilon}_{n+1}$, note L here is based on $\mathbf{u}^{n+1} + \delta\mathbf{u}^{k+1}$.
 - vi. Reset $\boldsymbol{\varepsilon}_{n+1}^p$ and $\boldsymbol{\alpha}^{n+1}$ to $\boldsymbol{\varepsilon}_{on+1}^p$ and $\boldsymbol{\alpha}_o^{n+1}$ (see Crisfield [2], pages 154 to 156, for discussion of why this is done).
 - vii. Use chosen plasticity algorithm here to update $\boldsymbol{\varepsilon}_{n+1}^p$, $\boldsymbol{\sigma}^{n+1}$, $\boldsymbol{\alpha}^{n+1}$ and \mathbf{C}_{ep}^{n+1} .
 - viii. Calculate the vector of new internal truss element axial forces \mathbf{N}_{n+1}^{k+1} . For truss element i the axial force is $(N_{n+1}^{k+1})_i = \sigma_{k+1}^{n+1} A_i$.
 - ix. Construct the vector of internal global forces \mathbf{F}_{int}^{n+1} based on \mathbf{N}_{n+1}^{k+1} .
 - x. Calculate the residual $\mathbf{R} = (\lambda^{n+1} + \delta\lambda_{k+1})\mathbf{F} - \mathbf{F}_{int}^{n+1}$ and modify the residual to account for the required supports.
 - xi. $R = \sqrt{\mathbf{R} \bullet \mathbf{R}}$
 - xii. Update iterations counter $k = k + 1$
- (o) **End** of while loop iterations
3. Update variables to their final value for the current increment
- $\lambda_{final}^{n+1} = \lambda_{(0)}^{n+1} + \delta\lambda_k$
 - $\mathbf{u}_{final}^{n+1} = \mathbf{u}_{(0)}^{n+1} + \delta\mathbf{u}^{(k)}$
4. **End Loop** over load increments

6.2 Single Truss Element - Monotonic Loading

A single truss element is pin supported at one end and is roller supported at the other end (Figure 3). The resulting truss element is monotonically loaded in tension along its axis in the direction of its only free displacement degree of freedom. The example element has $L = 60$ in., $A = 1.0$ in.² and $E = 29 \times 10^3$ ksi. A plot of load versus displacement is provided for three different plasticity models. In all cases the yield stress is taken as $\sigma_y = 36$ ksi. For linear isotropic hardening the plastic modulus $K = 500$ ksi. For the Ramberg Osgood [3] [5] model, $C = 10.7$ ksi and $m = 0.2$. For the Voce [7] model, $\sigma_y = 36$ ksi, $\sigma_u = 58$ ksi, $C = \sigma_u - \sigma_y$ and $\delta = 160$. The nonlinear analysis is achieved by a displacement control procedure. A maximum displacement of 0.5 inches is specified and is achieved by a series of 100 equal displacement increments. The results are presented in Figure 4.

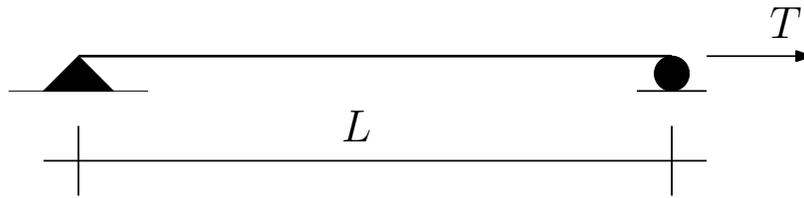


Figure 3: Single truss element loaded in tension.

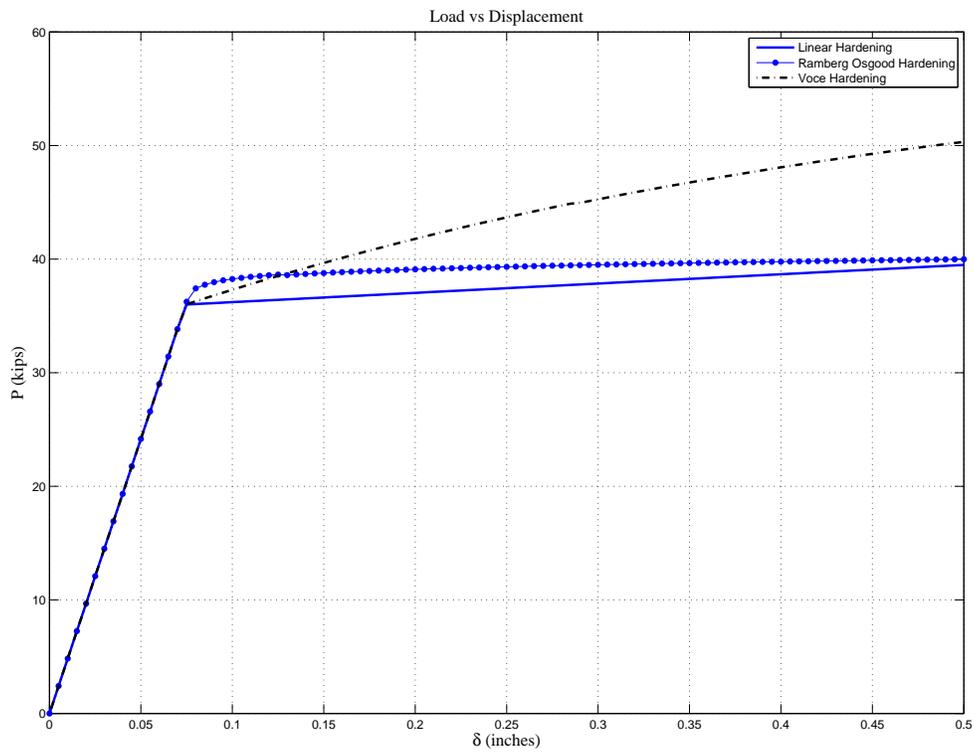


Figure 4: Single truss element monotonically loaded in tension with various 1D plasticity models.

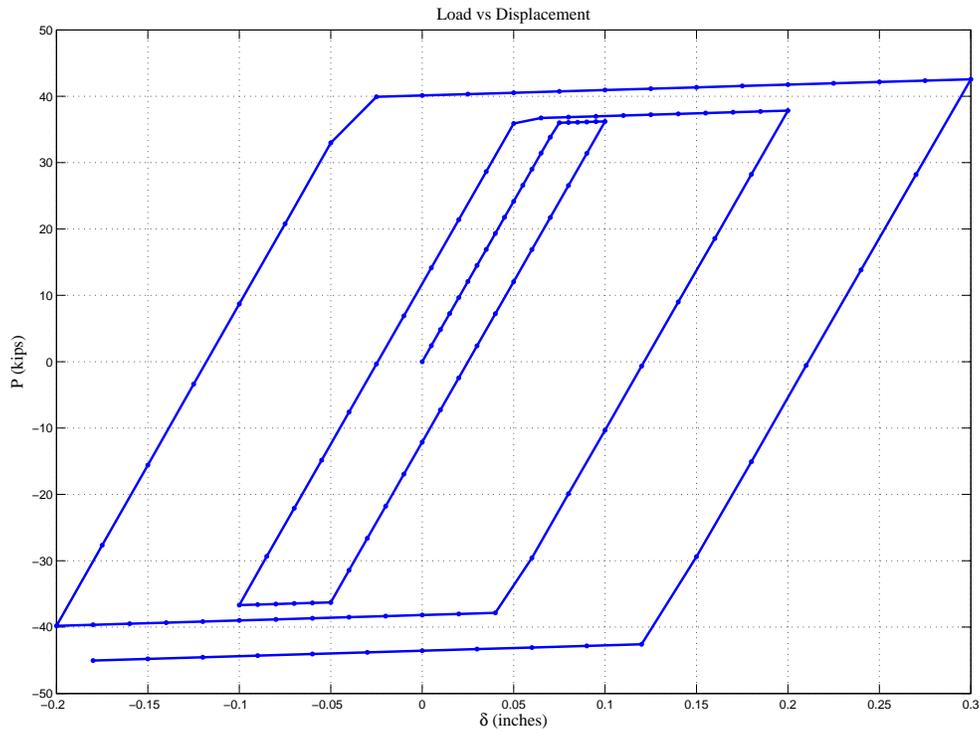


Figure 5: Single truss element cyclically loaded - Linear Isotropic Hardening

6.3 Single Truss Element - Cyclic Loading

A single truss element is pin supported at one end and is roller supported at the other end (Figure 3). The resulting truss element is cyclically loaded in tension and compression along its axis in the direction of its only free displacement degree of freedom. The example element has $L = 60$ in., $A = 1.0$ in.² and $E = 29 \times 10^3$ ksi. A plot of load versus displacement is provided for a linear isotropic hardening plasticity model. The yield stress is taken as $\sigma_y = 36$ ksi. The plastic modulus is $K = 500$ ksi. The implicit nonlinear analysis with Newton-Raphson iterations for equilibrium at the global level is achieved by a displacement control procedure for the cycles of displacement shown in Figure 5.

6.4 Cantilever Truss - Monotonic Loading

A cantilever truss is supported in the x direction at the bottom left node. It is pin supported at the top left node. The truss has 81 members and 42 nodes. The truss is 10 inches long and 0.5 inches tall. Each member has a cross-sectional area of 0.1 in.² and a modulus of elasticity of $E = 29 \times 10^3$ ksi. The truss is loaded (implicit nonlinear analysis) at its right end monotonically by a displacement control scheme to a maximum displacement of 0.7 inches in 200 equal increments. All truss members are modeled with the model by Voce [7]. A plot of load versus displacement is provided in Figure 6. The truss model is shown with its deflected shape in Figure 7.

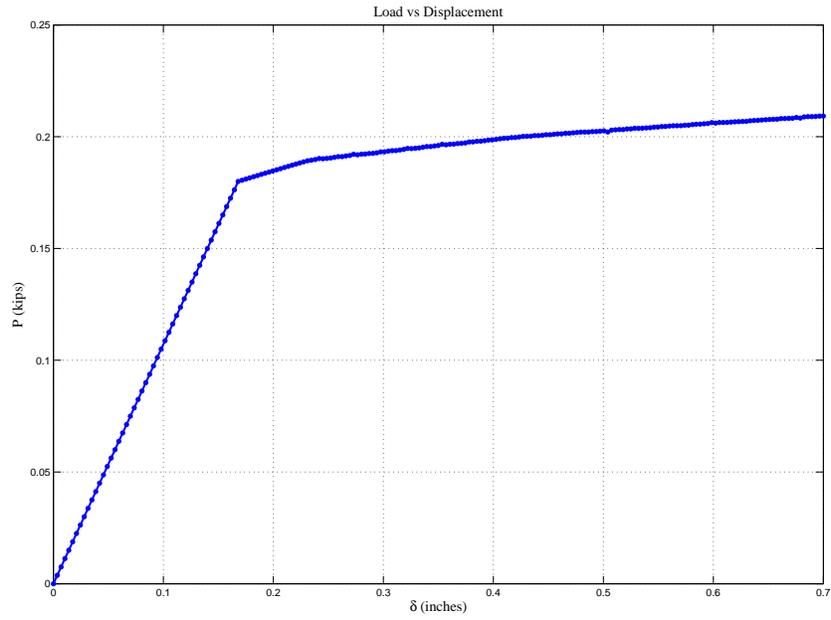


Figure 6: Cantilever Tip Load Versus Free End Vertical Displacement

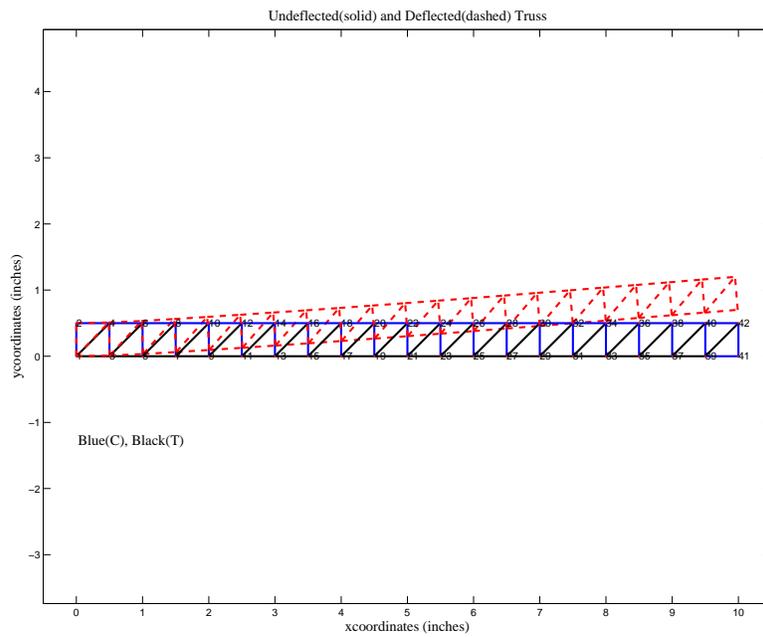


Figure 7: Original and Deflected Shape of Cantilever Truss

7 Conclusions

Plasticity models are often used in finite element analysis programs in order to account for nonlinear material behavior. In truss programs 1D plasticity is necessary to model element behavior when plastic material behavior is possible. The present paper describes the relevant terminology and provides numerous derivations and examples of 1D isotropic hardening plasticity models. An algorithm is provided which describes how such models are incorporated into a finite element truss program. Numerical results are provided for a single bar element undergoing monotonic loading and also a case for cyclic loading. Numerical results are also provided for the case of a cantilevered truss loaded to the point at which some members are beyond yielding. This introductory paper provides information germane toward taking the next step into 3D plasticity models.

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