Nonlinear Static - Explicit and Implicit Analysis Example
by Louie L. Yaw
Walla Walla University
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Introduction

This article presents an example nonlinear static problem. The problem is solved by both an explicit and implicit analysis. In both cases an incremental load control scheme is used. The example as presented is certainly contrived. To a beginner some aspects of this exercise may seem strange. However, the example problem is constructed to illustrate a variety of concepts that arise in a typical nonlinear finite element analysis. It is helpful to think of this exercise as providing some introductory tools that are useful to know for future problems.

The Problem

Consider the case of a simple bar in tension as shown in Figure 1. Suppose that the force in the bar is a nonlinear function of the displacement

$$f(u) = u + \frac{u^3}{3}. \tag{1}$$

From (1) it follows that the stiffness of the bar is

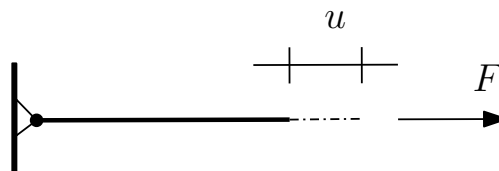$$k(u) = \frac{df}{du} = 1 + u^2. \tag{2}$$

Figure 1: Bar in tension.

Explicit Analysis

Clearly it is possible to plot a graph of force, f, versus displacement, u, by using equation (1) (see Figure 2). However, if the nonlinear force versus displacement relationship is not known, but the stiffness is known, it is possible to construct the graph of force versus displacement numerically. One method is to use an incremental explicit load control scheme. This is illustrated next. For this example three equal load increments (or steps) of $\Delta F = 1$ unit each are used to load the tension bar. The variables used are u for displacement, f for internal force in the bar, F for external force applied to the bar and k for stiffness. Incremental displacements or incremental externally applied forces are represented as Δu or ΔF, respectively. Use is made of the relationship $\Delta F = k\Delta u$. The analysis proceeds as follows:

Step 1

$$u_0 = 0.0, \ \Rightarrow k(u_0) = 1, \ \Delta F = 1, \ \Delta u_1 = \Delta F / k(u_0) = 1.0, \ u_1 = u_0 + \Delta u_1 = 1.0$$

Step 2

$$\Rightarrow k(u_1) = 1 + 1^2 = 2, \ \Delta F = 1, \ \Delta u_2 = \Delta F / k(u_1) = 1/2, \ u_2 = u_1 + \Delta u_2 = 1.5$$

Step 3

$$\Rightarrow k(u_2) = 1 + 1.5^2 = 3.25, \ \Delta F = 1, \ \Delta u_3 = \Delta F / k(u_2) = 1/3.25, \ u_3 = u_2 + \Delta u_3 \cong 1.81$$

Finally, from the three steps the following results are obtained.

$$F_{ext} = \Delta F + \Delta F + \Delta F = 3.0$$

$$f_{int} = u_3 + \frac{u_3^3}{3} = 1.81 + \frac{1.81^3}{3} \cong 3.79$$

It is evident from the results that external and internal forces are NOT in equilibrium because $F_{ext} \neq f_{int}$. Table 1 summarizes the main results for the explicit analysis.

Table 1: Summary of explicit analysis results

Step i	ΔF_i	Δu_i	u_i	$(F_{ext})_i$	$(f_{int})_i$	$f_{int} - F_{ext} = R$
1	1.0	1.000	1.00	1.0	1.333	0.333
2	1.0	0.500	1.50	2.0	2.625	0.625
3	1.0	0.308	1.81	3.0	3.790	0.790

Implicit Analysis

An implicit analysis is the same as the explicit analysis, except that at the end of each step Newton-Raphson iterations are used to enforce equilibrium before moving to the next step. Basically, an incremental force is applied to advance the solution forward at the beginning of a step. However, internal forces and external forces will not be in equilibrium unless the stiffness is linear for the given step. Hence, in order to achieve equilibrium, corrections must be made to the displacement. This is accomplished by using Newton-Raphson iterations to minimize the residual, $R(u) = f_{int} - F_{ext}$. Expanding the residual as a Taylor series about the current displacement u^j gives

$$R^{j+1} = R^j(u^j) + \frac{dR(u^j)}{du} \delta u^{j+1} + ... = R^j + k(u^j) \delta u^{j+1} + ... \tag{3}$$

By neglecting higher order terms in the series, setting $R^{j+1} = 0$ and solving for δu^{j+1} the following correction is obtained

$$\delta u^{j+1} = -[k(u^j)]^{-1}R^j. \tag{4}$$

Notice that the iteration variable is j and it may take several iterations for the norm of the residual, $\|R\|$, to be reduced below the chosen tolerance criteria (for the example below a tolerance of 10^{-2} is chosen). The implicit analysis proceeds as follows (results are approximate, they are more precise if done in a computer to machine precision):

Step 1

$$u_0 = 0.0, \Rightarrow k(u_0) = 1, \ \Delta F = 1, \ \Delta u_1 = \Delta F/k(u_0) = 1.0, \ u_1 = u_0 + \Delta u_1 = 1.0$$

Check the residual, $F_{ext} = \Delta F = 1$, $f_{int} = u_1 + u_1^3/3 = 4/3$, $R^{(0)} = f_{int} - F_{ext} = 1/3$
$\|R^{(0)}\| > 10^{-2} \Rightarrow$ Newton-Raphson iterations are necessary.

Calculate the correction to $u_1 = u_1^{(0)}$
$\delta u^{(1)} = -[k(u_1^{(0)})]^{-1}R^{(0)} = -2^{-1}(1/3) = -0.16667$.
$\Rightarrow$ The updated value of $u_1^{(1)} = u_1^{(0)} + \delta u^{(1)} = 1 + (-0.16667) = 0.83333$.

Check the residual again, $F_{ext} = 1$, $f_{int} = u_1^{(1)} + (u_1^{(1)})^3/3 = 1.02623$
$R^{(1)} = f_{int} - F_{ext} = 0.02623$
$\|R^{(1)}\| > 10^{-2} \Rightarrow$ another Newton-Raphson iteration is necessary.

Calculate the new total correction to $u_1 = u_1^{(0)}$
$\delta u^{(2)} = \delta u^{(1)} - [k(u_1^{(1)})]^{-1}R^{(1)} = -0.16667 - (1.6944^{-1})0.02623 = -0.1821$.

The updated value of $u_1^{(2)} = u_1^{(0)} + \delta u^{(2)} = 1 + (-0.18215) = 0.81785$.

Check the residual, $F_{ext} = 1$, $f_{int} = u_1^{(2)} + (u_1^{(2)})^3/3 = 1.000197$
$R^{(2)} = f_{int} - F_{ext} = 0.000197$
$\|R^{(2)}\| < 10^{-2} \Rightarrow$ no further iterations are necessary.

Therefore, the final value is $u_1 = u_1^{(2)} = 0.81785$.

Step 2
$$k(u_1) = 1 + 0.81785^2 = 1.66888, \ \Delta F = 1, \ \Delta u_2 = \Delta F/k(u_1) = 0.59920$$

$$u_2 = u_1 + \Delta u_2 = 1.41705$$

Check the residual, $F_{ext} = \Delta F = 1$, $f_{int} = u_2 + u_2^3/3 = 2.36554$
$R^{(0)} = f_{int} - F_{ext} = 0.36554$
$\|R^{(0)}\| > 10^{-2} \Rightarrow$ Newton-Raphson iterations are necessary.

Calculate the correction to $u_2 = u_2^{(0)}$
$\delta u^{(1)} = -[k(u_2^{(0)})]^{-1}R^{(0)} = -3.42508^{-1}(0.36554) = -0.10672$.
$\Rightarrow$ The updated value of $u_2^{(1)} = u_2^{(0)} + \delta u^{(1)} = 141705 + (-0.10672) = 1.31033$.

Check the residual again, $F_{ext} = 1$, $f_{int} = u_2^{(1)} + (u_2^{(1)})^3/3 = 2.060260$
$R^{(1)} = f_{int} - F_{ext} = 0.060260$
$\|R^{(1)}\| > 10^{-2} \Rightarrow$ another Newton-Raphson iteration is necessary.

Calculate the new total correction to $u_2 = u_2^{(0)}$
$\delta u^{(2)} = \delta u^{(1)} - [k(u_2^{(1)})]^{-1} R^{(1)} = -0.10672 - (3.02729^{-1})0.06026 = -0.12663.$

The updated value of $u_2^{(2)} = u_2^{(0)} + \delta u^{(2)} = 1.41705 + (-0.12663) = 1.29042.$

Check the residual, $F_{ext} = 1$, $f_{int} = u_2^{(2)} + (u_2^{(2)})^3/3 = 2.00663$
$R^{(2)} = f_{int} - F_{ext} = 0.00663$
$\|R^{(2)}\| < 10^{-2} \Rightarrow$ no further iterations are necessary.

Therefore, the final value is $u_2 = u_2^{(2)} = 1.29042.$

**Step 3** is completed similar to steps 1 and 2, the final value is $u_3 = 1.6097$.

The results of the implicit analysis are summarized in Table 2. It is evident from the table that the external forces and internal forces are coming into equilibrium.

Table 2: Summary of implicit analysis results

Step i	ΔF_i	u_i	$(F_{ext})_i$	$(f_{int})_i$	$f_{int} - F_{ext} = R$
1	1.0	0.81785	1.0	1.000197	0.000197
2	1.0	1.29042	2.0	2.00663	0.00663
3	1.0	1.61	3.0	3.001	0.001

Plot of results

A plot of the results is shown in Figure 2a. It is evident from the plot that the explicit analysis drifts from the exact solution. If 20 increments are used the explicit analysis more closely follows the exact solution as shown in Figure 2b. However, even with more increments the explicit analysis still drifts from the exact solution. The drift from the exact solution illustrates the lack of equilibrium between the internal and external forces. To correct for this problem an implicit analysis is required. The results for the implicit analysis are shown in Figure 2a. The Newton-Raphson iterations correct the incremental steps so that they land on the exact solution according to the specified tolerance. Excellent agreement is achieved with just 3 increments and in each step only two Newton-Raphson iterations are required. The explicit and implicit analyses are completed by using simple MATLAB scripts (`non_e.m` and `non_i.m`).
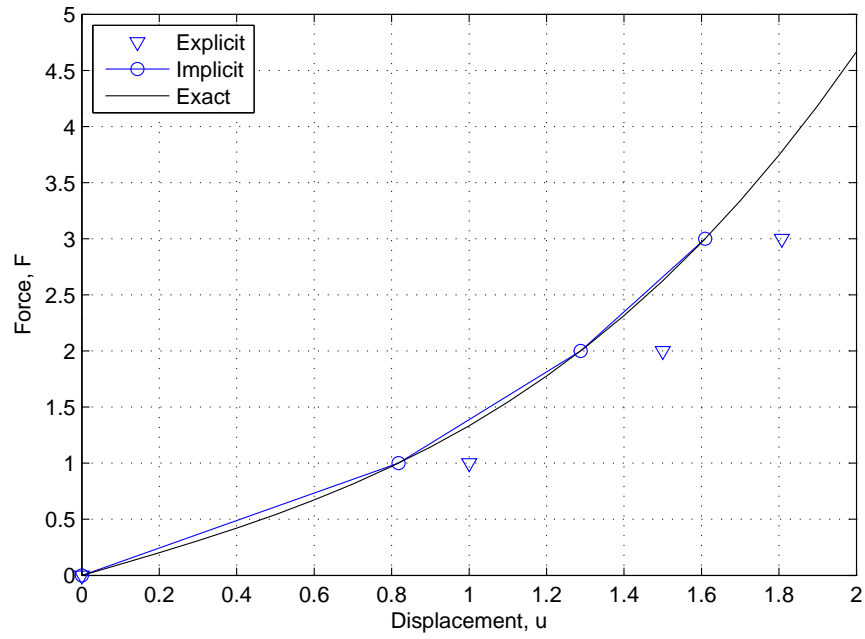
(a)
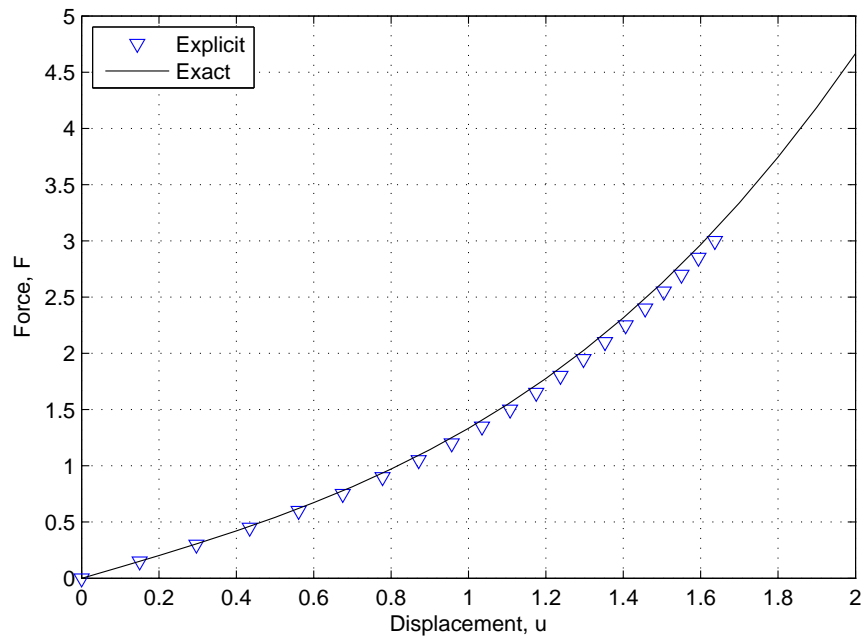
(b)

Figure 2: Single bar in tension: (a) Comparison of explicit, implicit and exact results of load versus displacement; (b) explicit results using 20 increments.

Discussion

In practice a more sophisticated convergence criteria may be necessary when solving complex structures with many degrees of freedom. However, for this example problem minimization of the norm of the residual is a sufficient criteria. It is important to note how the initial increment due to the incremental load advances the solution forward. Then if an implicit analysis is employed, Newton-Raphson iterations are used to make corrections to the displacement in order to achieve equilibrium. Once satisfactory equilibrium is achieved, a new load increment is applied to advance the solution and the process is repeated. Note that an exact expression for force versus displacement is not always known. More discussion later such as which is better explicit or implicit? When to use explicit instead of implicit? How does this example relate to nonlinear structure problems with many degrees of freedom? What increment size should be used in explicit? What increment size should be used in implicit? Increment size may affect convergence rate. If increment size is appropriate the Newton-Raphson iterations converge quadratically. Note how the correction δu keeps being updated in the Newton-Raphson iterations. For the given step the original estimated displacement is updated with the final δu only after the equilibrium criteria is satisfied. What tolerance is required to enforce equilibrium? More later...

Conclusion

A nonlinear analysis is illustrated using an incremental explicit and implicit procedure. The results are plotted for comparison. It is evident from the results that the explicit analysis drifts from the true solution. To overcome this problem an implicit analysis is used, which includes Newton-Raphson iterations to enforce equilibrium between internal and external forces. The techniques demonstrated are for a load control scheme. For other schemes such as displacement control, arc length control and generalized displacement control the reader is referred to the references given below. Many of the concepts and methods introduced are similar to concepts and methods used in a nonlinear finite element analysis. Therefore, the given example is a useful learning tool for students learning about methods of solution for nonlinear problems.

References

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