Undamped Vibration of a Beam

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Problem - Undamped Transverse Beam Vibration

\[ p(x, t) + u_m(x), EI(x) \]

\[ +L \]

\[ +x \]

\[ 0 \]

\[ +u \]

\[ m(x), EI(x) \]

\[ dx \]

\[ V + \frac{\partial V}{\partial x} dx \]

\[ V \]

\[ \epsilon_1 dx \]

\[ \epsilon_2 dx \]

\[ M + \frac{\partial M}{\partial x} dx \]

\[ f_I = mdx \frac{\partial^2 u}{\partial t^2} \]

Inertial Force by D’Alembert’s Principle
Derivation of PDE

- Sum Forces Vertically, choosing + up

\[ V - \left( V + \frac{\partial V}{\partial x} \, dx \right) + p(x, t) \, dx - m(x) \, dx \frac{\partial^2 u}{\partial t^2} = 0 \quad (1) \]

- Sum Moments about 0, choosing CCW as + rotation

\[ -M - V \, dx + p(x, t) \epsilon_1 \, dx^2 + m(x) \epsilon_2 \, dx^2 \frac{\partial^2 u}{\partial t^2} + M + \frac{\partial M}{\partial x} \, dx = 0 \quad (2) \]

Simplifying (1), and in (2) ignoring higher order terms in the limit as \( dx \to 0 \) gives

\[ \frac{\partial V}{\partial x} = p(x, t) - m(x) \frac{\partial^2 u}{\partial t^2}, \quad \text{and} \quad V = \frac{\partial M}{\partial x} \quad (3) \]
From mechanics of materials class, moment curvature relation (given here to save time)

\[ M = EI(x) \frac{\partial^2 u}{\partial x^2} \]  

Substituting equation two of (3) and equation (4) into equation one of (3) and rearranging yields

\[ m(x) \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left[ EI(x) \frac{\partial^2 u}{\partial x^2} \right] = p(x, t) \]  

Equation (5) is the PDE governing the motion \( u(x, t) \), subject to the external forcing function \( p(x, t) \).
Solving the PDE

- Analytical solution difficult or impossible to obtain due to $m(x)$ and $I(x)$.
- Numerical methods such as Finite Element Method or Finite Differences can solve the PDE.
- Can simplify the PDE to demonstrate analytical methods by the following assumptions:
  - $m(x) = m =$ constant along the beam length
  - $I(x) = I =$ constant along the beam length
  - $p(x, t) = 0$, ie, no forcing function
After simplifying assumptions the governing PDE (5) becomes

\[ m \frac{\partial^2 u}{\partial t^2} + EI \frac{\partial^4 u}{\partial x^4} = 0 \]  

(6)

To make things pretty at the end, define \( a^2 = EI/m \), so that

\[ \frac{\partial^2 u}{\partial t^2} + a^2 \frac{\partial^4 u}{\partial x^4} = 0 \]  

(7)
Assume a solution of the following form

\[ u(x, t) = \phi(x)q(t) \]  \hspace{1cm} (8)

Substitute (8) into the PDE (7) to get

\[ \phi \frac{\partial^2 q}{\partial t^2} + a^2 q \frac{\partial^4 \phi}{\partial x^4} = 0 \]  \hspace{1cm} (9)

By separation of variables, observe that l.h.s and r.h.s must equal a constant, \( \beta^4 \)

\[ - \frac{1}{a^2 q} \frac{\partial^2 q}{\partial t^2} = \frac{1}{\phi} \frac{\partial^4 \phi}{\partial x^4} = \beta^4 \]  \hspace{1cm} (10)
Solving the PDE

- From (10), two ODE’s are obtained

\[
\frac{\partial^4 \phi}{\partial x^4} - \beta^4 \phi = 0 \quad (11)
\]

\[
\frac{\partial^2 q}{\partial t^2} + \beta^4 a^2 q = 0 \quad (12)
\]

- The respective solutions are

\[
\phi(x) = A \sinh \beta x + B \cosh \beta x + C \sin \beta x + D \cos \beta x \quad (13)
\]

\[
q(t) = E \sin a\beta^2 t + F \cos a\beta^2 t \quad (14)
\]

- ∴ solution of the PDE (6) is \( u(x, t) = \phi(x)q(t) \).
To solve a realistic problem, boundary conditions must be specified.

The six boundary conditions (BC’s) are:

1. \( @ x = 0, \ u(0, t) = \phi(0)q(t) = 0 \)
2. \( @ x = L, \ u(L, t) = \phi(L)q(t) = 0 \)
3. \( @ x = 0, \ u''(0, t) = \phi''(0)q(t) = 0 \)
4. \( @ x = L, \ u''(L, t) = \phi''(L)q(t) = 0 \)
5. \( @ t = 0, \ \dot{u}(x, 0) = \phi(x)\dot{q}(0) = 0 \)
6. \( @ t = 0, \ u(x, 0) = Gx(L - x), \ G \text{ specified constant} \)
Applying the boundary conditions to $\phi(x)$

Applying the first four boundary conditions yield the following results

1. $\phi(0) = B + D = 0$
2. $\phi(L) = A \sinh \beta L + B \cosh \beta L + C \sin \beta L + D \cos \beta L = 0$
3. $\phi''(0) = B \beta^2 - D \beta^2 = 0 \quad \Rightarrow \quad B - D = 0$
4. $\phi''(L) = A \beta^2 \sinh \beta L + B \beta^2 \cosh \beta L - C \beta^2 \sin \beta L - D \beta^2 \cos \beta L = 0$

From BC’s (1) and (3), $B = 0$ and hence $D = 0$
Applying the boundary conditions to $\phi(x)$

- From BC's (2) and (4)
  \[ A \sinh \beta L + C \sin \beta L = 0 \]
  \[ A \sinh \beta L - C \sin \beta L = 0 \]

- The above results imply
  \[ A \sinh \beta L = 0 \quad \text{and} \quad C \sin \beta L = 0 \quad (15) \]

- From the first expression of (15), $A = 0$. If $A = 0$ is not chosen, $\beta = 0$ is required and this leads to $\phi(x) = 0$ for all $x$ which is the at rest condition (not very interesting).

- Using the remaining case (since $A = 0$), either $C = 0$ or $\sin \beta L = 0$. Choosing $C = 0$ isn’t an option since that leads to $\phi(x) = 0$ for all $x$ which is the uninteresting at rest condition.
Applying the boundary conditions to $\phi(x)$

- Therefore, must have $\sin \beta L = 0$, which implies $\beta L = n\pi$.
- After solving for $\beta$, the $n$ solutions (which satisfy the B.C’s) for $\phi(x)$ are
  \[ \phi_n(x) = C_n \sin \frac{n\pi x}{L} \] (16)
- This implies that the beam vibrates in the following natural mode shapes for $n = 1, 2, 3, 4...$
Applying BC (5) yields

\[ \dot{q}(0) = -a\beta^2 E \sin a\beta^2 0 + a\beta^2 F \cos a\beta^2 0 = 0 \quad (17) \]

The sine term equals zero and hence \( F = 0 \). As a result

\[ q(t) = E \cos a\beta^2 t \quad (18) \]

In light of the fact that \( \beta = n\pi/L \)

\[ q_n(t) = E_n \cos \frac{an^2\pi^2 t}{L^2} \quad (19) \]
Combining (16) and (19) and defining $b_n = C_n E_n$ yields

$$u_n(x, t) = \phi_n(x) q_n(t) = b_n \sin \frac{n\pi x}{L} \cos \frac{a n^2 \pi^2 t}{L^2}$$  \hspace{1cm} (20)$$

Equation (20) satisfies the PDE and the first 5 BC's for any value of $n$ and arbitrary constants $b_n$. As a result, any linear combination of (20) also satisfies the requirements so that

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \cos \frac{a n^2 \pi^2 t}{L^2}$$  \hspace{1cm} (21)$$
Applying the boundary condition (6)

- To satisfy BC (6) the following must be true

\[ u(x, 0) = Gx(L - x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad (22) \]

- Hence, the \( b_n \) are the sine Fourier coefficients for \( Gx(L - x) \). That is

\[ b_n = \frac{2}{L} \int_{0}^{L} Gx(L - x) \sin \frac{n\pi x}{L} \, dx \quad (23) \]

\[ = \frac{8GL^2}{n^3\pi^3} \quad \text{for } n \text{ odd} \quad (24) \]

\[ = 0 \quad \text{for } n \text{ even} \quad (25) \]
Final solution of the BVP

Using the results of (21) and (23) gives the final solution of the BVP.

\[ u(x, t) = \sum_{n=1,3,5,...}^{\infty} \frac{8GL^2}{n^3\pi^3} \sin \frac{n\pi x}{L} \cos \frac{an^2\pi^2 t}{L^2} \]  

(26)

Comments:
- Recall \( a^2 = EI/m \) which is known
- \( G \) specifies initial amplitude at \( t = 0 \), hence is known
- By observing the cosine term of (26) it is concluded that the natural frequencies for the beam are

\[ \omega_n = \frac{n^2\pi^2}{L^2} \sqrt{\frac{EI}{m}} \]