Applications of Eigenvalues & Eigenvectors

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For Linear Algebra Class
November 17, 2014

Outline

1. The eigenvalue/eigenvector problem
2. Principal stresses and directions in stress analysis
3. Fundamental frequencies and mode shapes in vibrations
4. Critical loads and buckled shapes in buckling analysis
5. Applications in electrical area/ feedback and control
6. Principal mass moment of inertia in 3D

1. The eigenvalue/eigenvector problem

Find vector **v** and scalar λ that satisfies the following:

$$\mathbf{Kv} = \lambda \mathbf{v} \tag{1}$$

where,
$\mathbf{K}$ = nxn matrix
$\mathbf{v}$ = nx1 vector, an eigenvector
λ= a scalar, an eigenvalue

n vector and scalar pairs will satisfy equation (1). How to find such pairs is a linear algebra problem.

Equation (1) arises on occasion during the solution of various types of engineering problems

1. The eigenvalue/eigenvector problem

An example (3x3 matrix $\Rightarrow$ 3 eigenvalues and 3 eigenvectors):

$$\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \tau_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \tau \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} 10 & 8 & 0 \\ 8 & 14 & 5 \\ 0 & 5 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \tau \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{3}$$

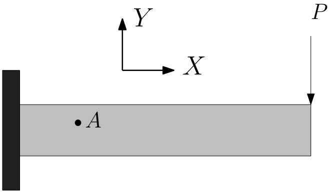
MATLAB solution to eigenvalue problem:

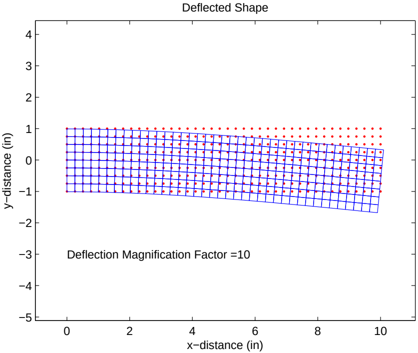
$$\tau_1 = 2.17, \mathbf{v} = \begin{bmatrix} -0.64 \\ 0.62 \\ -0.46 \end{bmatrix}, \quad \tau_2 = 9.29, \mathbf{v} = \begin{bmatrix} 0.55 \\ -0.05 \\ -0.83 \end{bmatrix}, \tag{4}$$

$$\tau_3 = 21.54, \mathbf{v} = \begin{bmatrix} -0.54 \\ -0.78 \\ -0.31 \end{bmatrix}$$

2. Principal stresses and directions in stress analysis

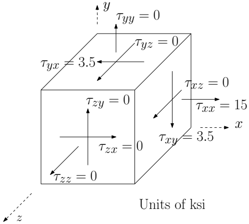
Consider a two dimension problem. A cantilever beam loaded at its free end.





2. Principal stresses and directions in stress analysis

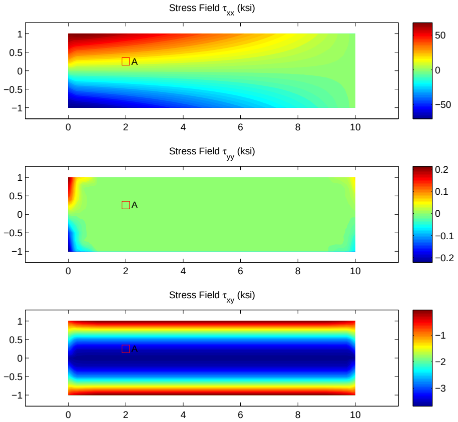
Stress block from point A



Units of ksi

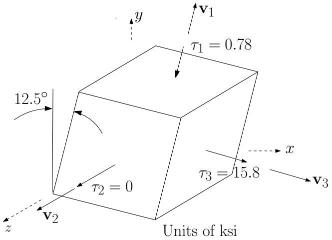
$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \tau_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix}$$

$$\boldsymbol{\tau} = \begin{bmatrix} 15 & -3.5 & 0 \\ -3.5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



2. Principal stresses and directions in stress analysis

MATLAB solution to eigen problem:



$$\tau_1 = -0.78, \mathbf{v}_1 = \begin{bmatrix} 0.2166 \\ 0.9763 \\ 0.0000 \end{bmatrix},$$

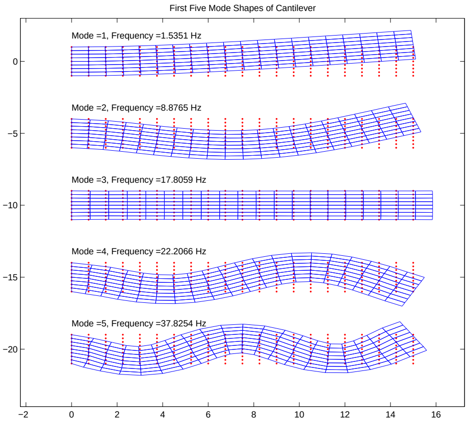
$$\tau_2 = 0.00, \mathbf{v}_2 = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 1.0000 \end{bmatrix},$$

$$\tau_3 = 15.8, \mathbf{v}_3 = \begin{bmatrix} 0.9763 \\ -0.2166 \\ 0.0000 \end{bmatrix}$$

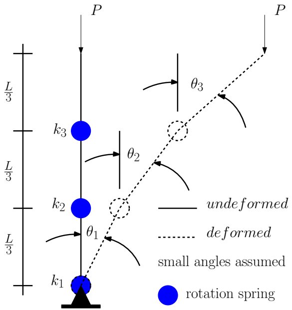
3. Fundamental frequencies and mode shapes in vibrations

Numerical Eigen problem solution

- 2D FEA of cantilever
- x & y dofs at 189 nodes (2 x 189 = 378 dofs)
- $\Rightarrow$ 378 eigenvalues & eigenvectors
- $\Rightarrow$ 378 frequencies & mode shapes
- Eigenvectors are 378th dimensional vectors!
- $\bar{\mathbf{K}}\mathbf{v} = \lambda\mathbf{v}$, $\bar{\mathbf{K}} = 378x378$ in this example!
- Finer Grid=More accurate



4. Critical loads & buckled shapes in buckling analysis



A deformed shape equilibrium analysis results in the following:

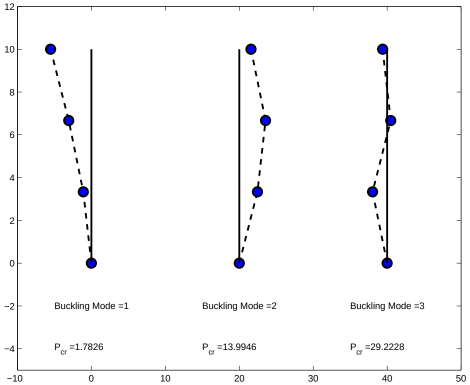
$$\begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k3 \\ 0 & -k_3 & k3 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \frac{PL}{3} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$\mathbf{K}\boldsymbol{\theta} = \lambda\boldsymbol{\theta}$$

4. Critical loads & buckled shapes in buckling analysis

For the case of
$L = 10$ inches
$k_1 = 30$ kip-in/rad
$k_2 = 30$ kip-in/rad
$k_3 = 30$ kip-in/rad

Eigenvalue $= \frac{P_{cr} L}{3}$
Eigenvector $\Rightarrow$ shape



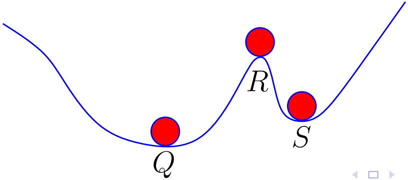
5. Applications in electrical engineering - feedback and control

Outline of conceptual feedback and control

- Model dynamic system such as airplane, car, rocket
- $\mathbf{M}\ddot{\phi} + \mathbf{C}\dot{\phi} + \mathbf{K}\phi = \mathbf{F}(t)$
- The mathematical model of the system has inherent eigenvalues and eigenvectors
- Eigenvalues describe resonant frequencies where the system will have its largest, often excessive, response.
- We can choose $\mathbf{F}(t)$ to reduce the system response at the resonant frequencies.

5. Applications in electrical engineering - feedback and control

- Perhaps let $\mathbf{F}(t) = \mathbf{A}\dot{\phi} + \mathbf{B}\phi$ and insert into dynamic system model
- The new system is $\mathbf{M}\ddot{\phi} + \bar{\mathbf{C}}\dot{\phi} + \bar{\mathbf{K}}\phi = \mathbf{0}$
- where $\bar{\mathbf{C}} = \mathbf{C} - \mathbf{A} =$ velocity dependent damping
- and $\bar{\mathbf{K}} = \mathbf{K} - \mathbf{B} =$ displacement dependent forcing
- We can adjust **A** and **B** so that the eigenvalues of the new 'barred' system are different from the original system
- By doing so we cause (control) the system to avoid excessive vibration or instability



6. Principal mass moment of inertia in 3D

3D kinetics of a rigid body

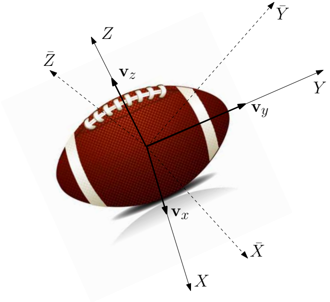
Inertia tensor (components dependent on $\bar{X}\bar{Y}\bar{Z}$ coordinate axes orientation)

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

It is possible to orient the axes such that

$$I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

6. Principal mass moment of inertia in 3D



In $\bar{X}\bar{Y}\bar{Z}$ coordinate system

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

I_x, I_y, I_z are eigenvalues

$\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z$ are eigenvectors wrt $\bar{X}\bar{Y}\bar{Z}$

In XYZ coordinates

$$I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

Conclusion

A few comments:

- Many applications of eigenvalues and eigenvectors in engineering
- **K** is not always symmetric
- eigenvalues are not always positive or real
- eigenvectors are orthogonal
- eigenvalues are invariant wrt to choice of coordinate axes

Questions.

$$\mathbf{K}\theta =?\theta$$