A Co-rotational Virtual Element Method for 2D Elasticity and Plasticity

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Acknowledgments:

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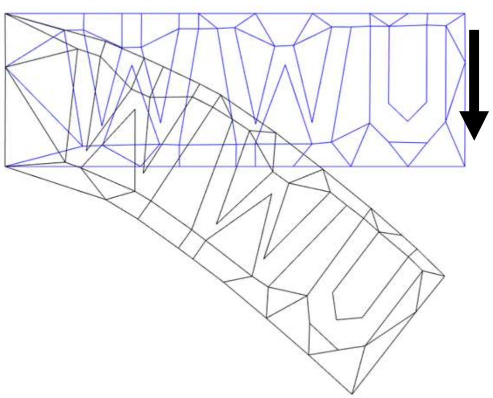
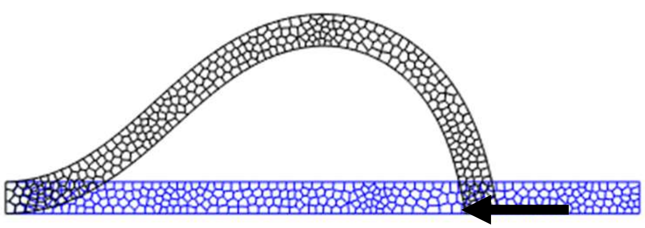
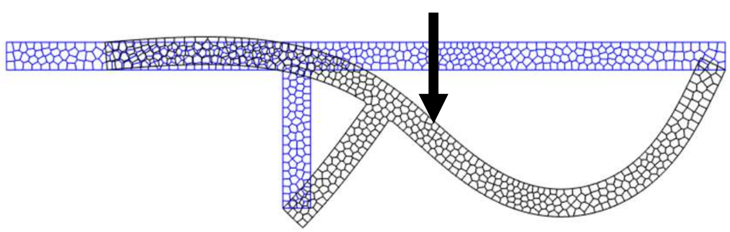
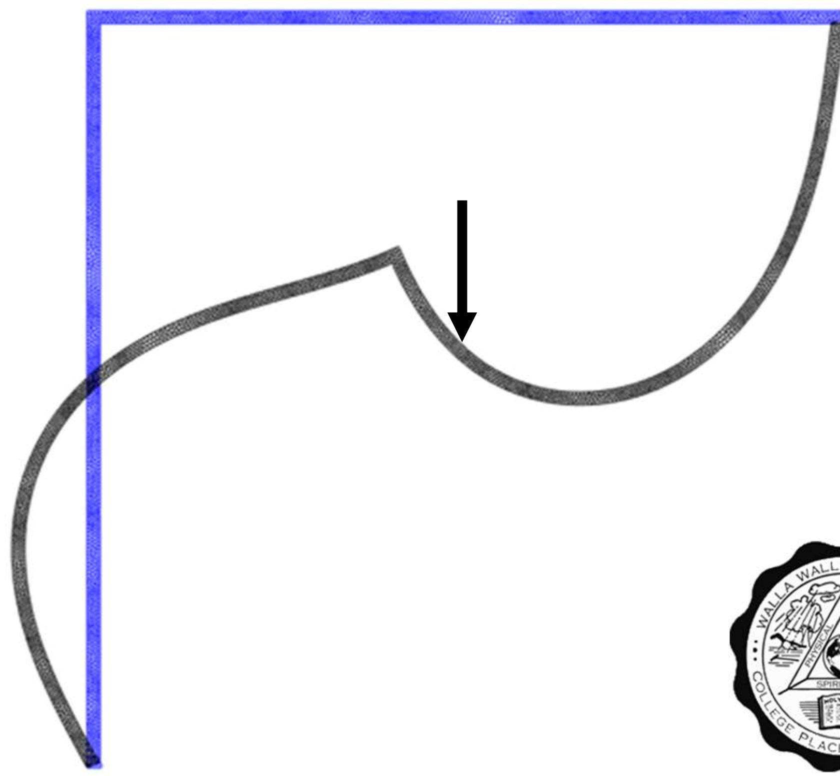






Outline

1. Motivation: Large Displacements and Rotations with VEM
2. Virtual Element Method (VEM)
3. Co-rotational VEM – Elasticity
4. Co-rotational VEM – Plasticity
5. Numerical Implementation
6. Numerical Examples
7. Conclusions
8. Questions



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- Post-collapse behavior
- With path following, identify local and global maximums

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Co-rotational Formulation

Pros:

- Avoid alternative stress and strain measures (objectivity maintained)
- Small strain plasticity easily incorporated
- Achieve geometric nonlinearity

Cons:

- Limited to small to moderate strains (extension to large strains is possible)

2. Virtual Element Method (VEM)

- 2D Elasto-statics problem solved by VEM
  - Find $\mathbf{u} \in \boldsymbol{\mathcal{V}}$ such that

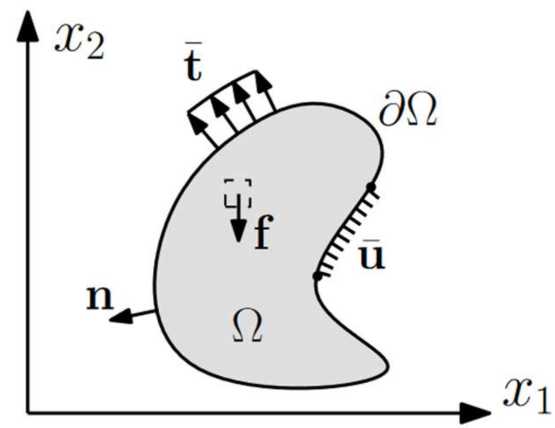
$$a(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in \boldsymbol{\mathcal{V}},$$

  - where

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v}) d\Omega,$$

  - and

$$L(\mathbf{v}) = \int_\Omega \mathbf{v} \cdot \mathbf{f} d\Omega + \int_{\partial\Omega_t} \mathbf{v} \cdot \bar{\mathbf{t}} d\partial\Omega.$$



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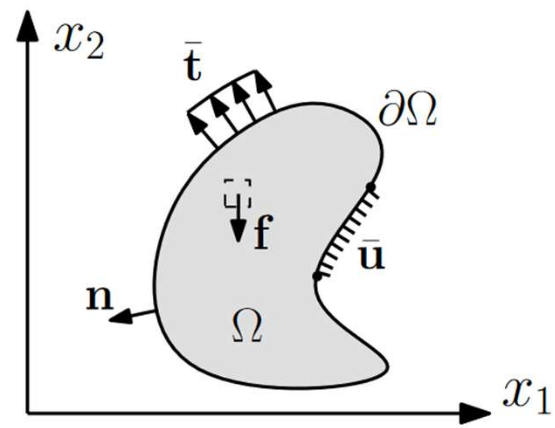
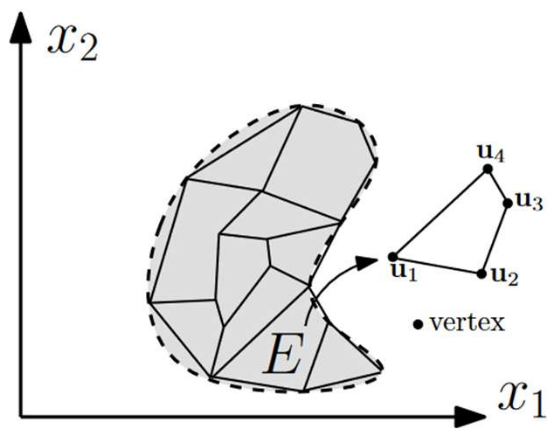


- Discretize domain with arbitrary polygons



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- Current work uses first order 2D polynomial basis

$$\mathbf{P}_1 = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\eta \\ \xi \end{pmatrix}, \begin{pmatrix} \eta \\ \xi \end{pmatrix}, \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \eta \end{pmatrix} \right]$$

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- Stresses and strains constant over each virtual element

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- VEM element stiffness matrix (consistency part and stabilization part)

$$\mathbf{k}_E = \mathbf{k}_E^c + \mathbf{k}_E^s$$

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Plane Stress

$$\mathbf{C} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

Plane Strain

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- and $\mathbf{S}_E^d$ a diagonal matrix with diagonal terms taken as

$$(\mathbf{S}_E^d)_{ii} = \max(\alpha_0 \ \mathrm{tr}(\mathbf{C})/m, (\mathbf{k}_E^c)_{ii})$$

$$\alpha_0 = 1$$
$$m = 3 \ for \ 2D$$

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3. Co-rotational VEM - Elasticity

Basics of Co-rotational – Local co-rotating frame

- Current Co-rotational frame associated with each virtual element

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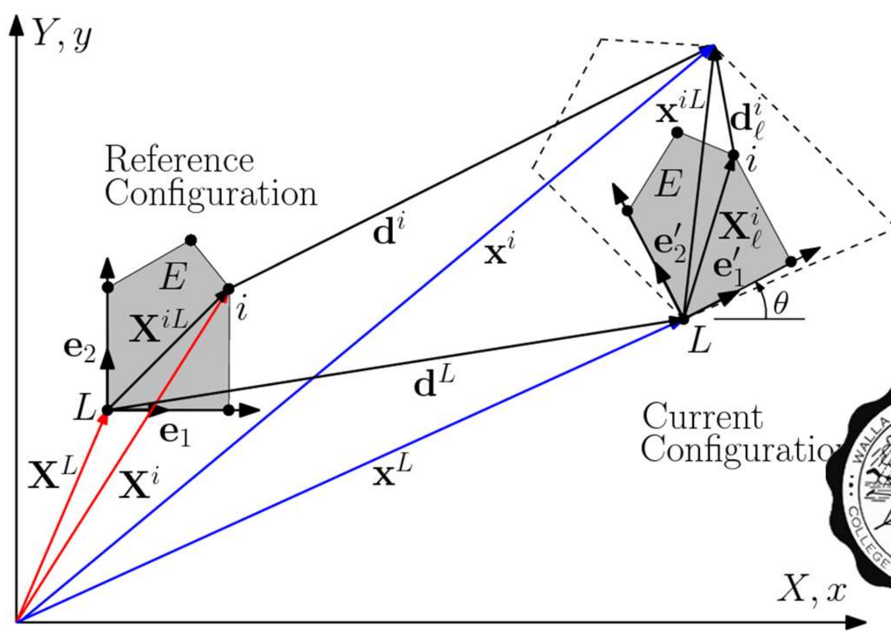

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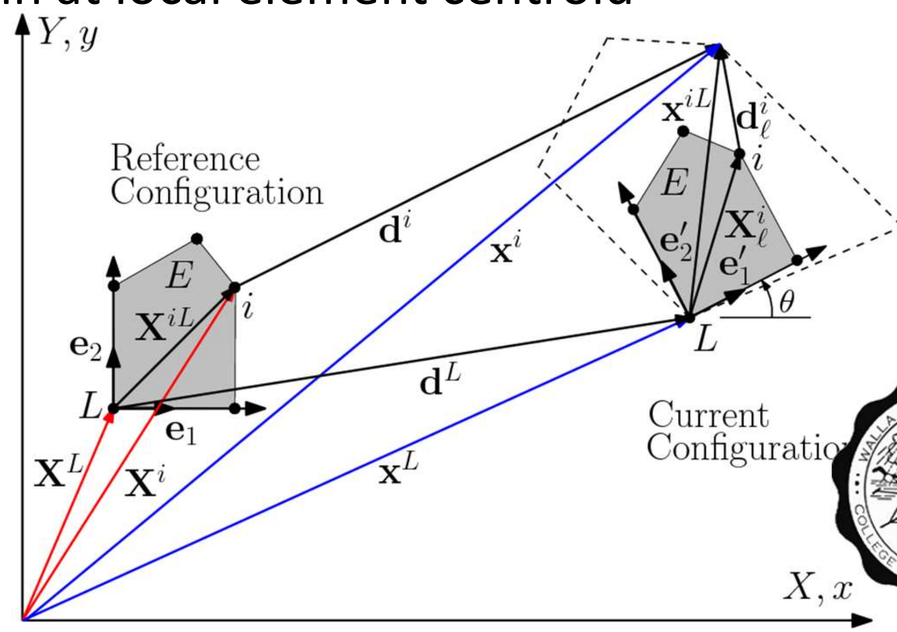

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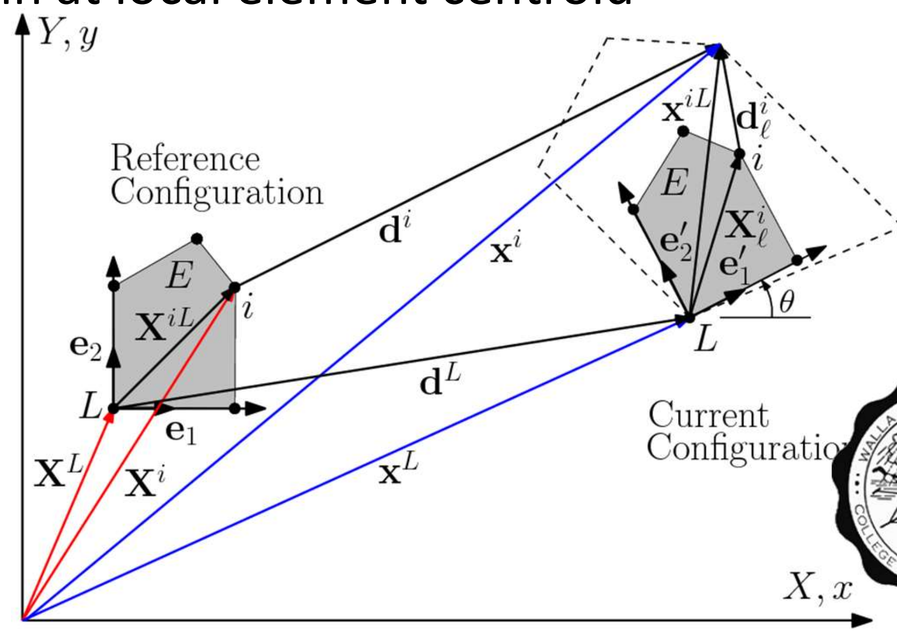

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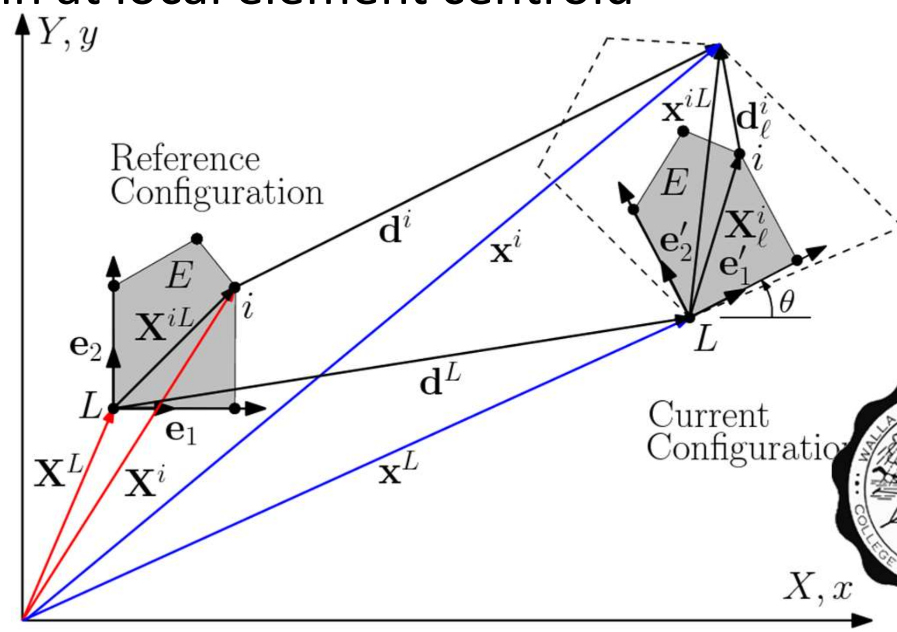
- Formulation accomplished by
  - Initial nodal coordinates
  - Current nodal displacements
  - Shape function derivatives



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Basics of Co-rotational

- Co-rotational Nonlinear Analysis Requires
  a. Global Element Tangent Stiffness Matrix

$$\mathbf{k}_T = \mathbf{T}^T \mathbf{k}_{E\ell} \mathbf{T} + \mathbf{k}_{t\sigma} \quad \begin{cases} \mathbf{T} = \text{Transformation matrix} \\ \mathbf{k}_{E\ell} = \mathbf{k}_E^c + \mathbf{k}_E^s \quad \text{Local VE stiffness} \\ \mathbf{k}_{t\sigma} = \text{Initial or geometric stiffness} \end{cases}$$

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- Transformation provides relations between local and global quantities

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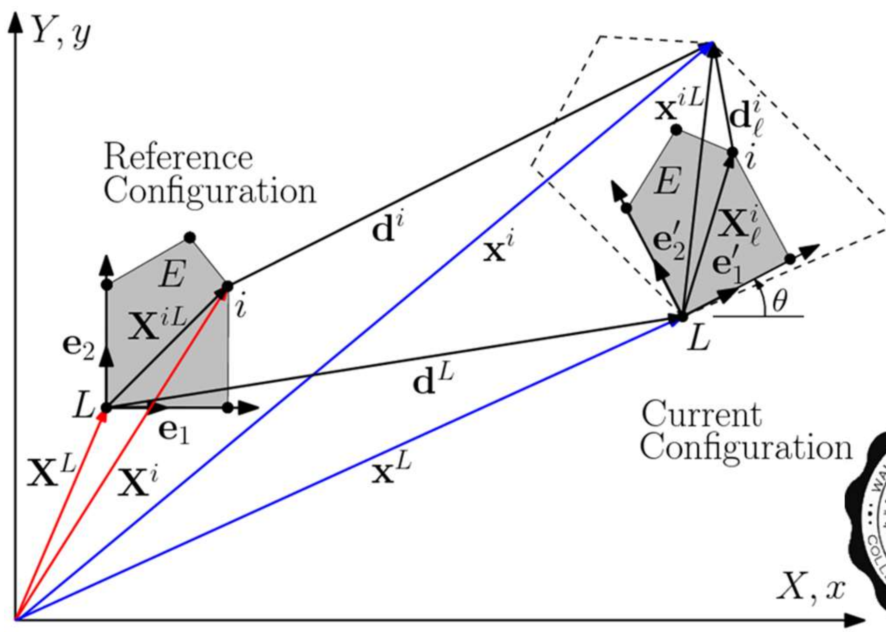

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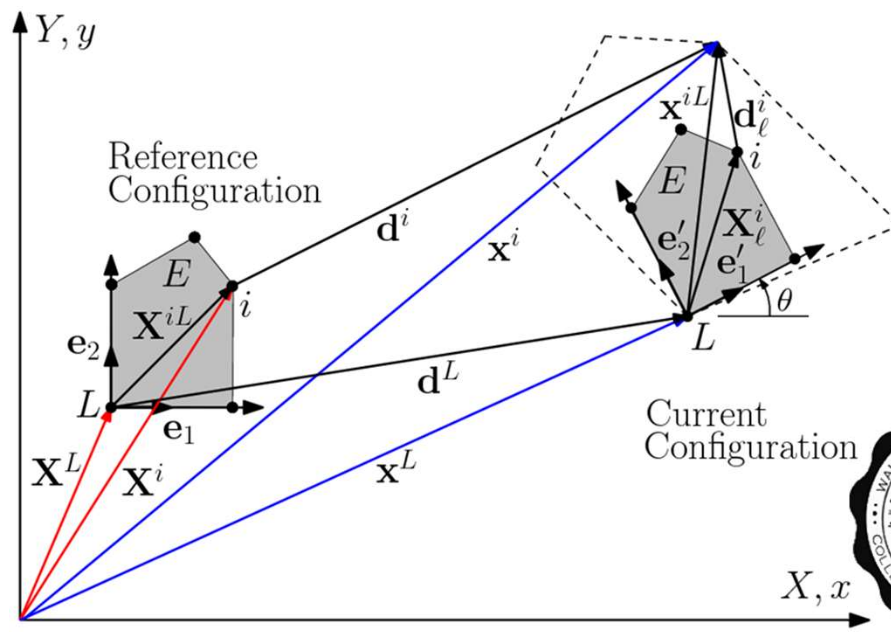
- Transformation found by taking
  - Variation of $\mathbf{d}_\ell^i = \mathbf{Q}^T\mathbf{x}^{iL} - \mathbf{X}^{iL}$

$$\mathbf{x}^{iL} = \mathbf{x}^i - \mathbf{x}^L$$

$$\Rightarrow \ \delta\mathbf{d}_\ell^i = \mathbf{Q}^T\delta\mathbf{x}^{iL} + \delta\mathbf{Q}^T\mathbf{x}^{iL}$$



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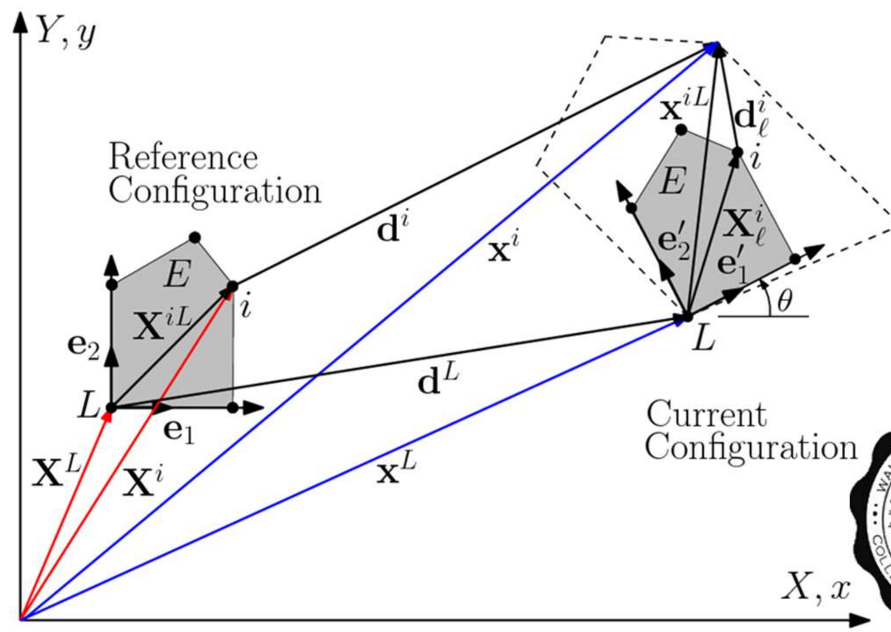
  - Leading to

$$\mathbf{T} = \bar{\mathbf{Q}} + \bar{\mathbf{x}}_\ell \mathbf{v}^T$$



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3. Co-rotational VEM - Elasticity

Basics of Co-rotational – Geometric Stiffness

- The PVW reveals initial stiffness related to variation of transformation matrix and local internal force vector

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- After a lengthy derivation one finds

$$\mathbf{k}_{t\sigma} = \sum_{j=1}^{2n}\mathbf{q}_{E\ell}^{j}\mathbf{G}^{j}$$

3. Co-rotational VEM - Elasticity

Basics of Co-rotational – Geometric Stiffness

- The PVW reveals initial stiffness related to variation of transformation matrix and local internal force vector

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- $\mathbf{G}^j$ contains - co-rotating basis, current coords, shape func derivatives
- Initial stiffness, $\mathbf{k}_{t\sigma}$, is crucial for quadratic convergence of NR-iterations.

4. Co-rotational VEM - Plasticity

2D small strain plasticity formulation easily included

- Simply insert elasto-plastic modular matrix into consistency term

$$\mathbf{k}_E^c = \int_E (\mathbf{B}\tilde{\mathbf{\Pi}})^T \mathbf{C}_{ep} (\mathbf{B}\tilde{\mathbf{\Pi}}) dE,$$

J. C. Simo and R. L. Taylor, A return mapping algorithm for plane stress elastoplasticity, *International Journal for Numerical Methods in Engineering*, 22:649–670, 1986.

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- Plane stress formulation by Simo and Taylor

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5. Numerical Implementation

- Implicit Newton-Raphson-iterations to enforce global equilibrium

M. S. Floater, Generalized barycentric coordinates and applications, *Acta Numerica* (2016).

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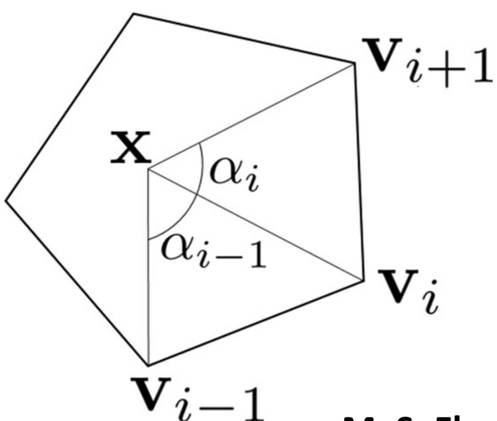
5. Numerical Implementation

- Implicit Newton-Raphson-iterations to enforce global equilibrium
- Arc-length method used for path following scheme
- Shape function derivatives needed at virtual element centroid (need alternative since VEM shape functions not known at element centroids)
  - ➢Mean Value Coordinates (used in this work, least costly option)



$$\phi_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{j=1}^{n} w_j(\mathbf{x})} \qquad w_i(\mathbf{x}) = \frac{\tan(\alpha_{i-1}/2)+\tan(\alpha_i/2)}{\|\mathbf{v}_i - \mathbf{x}\|}$$

$$\nabla\phi_i = (\mathbf{R}_i - \textstyle\sum_{j=1}^{n} \phi_j \mathbf{R}_j)\phi_i$$, etc.

M. S. Floater, Generalized barycentric coordinates and applications, *Acta Numerica* (2016).

6. Numerical Examples

Linear Elastic Cantilever

Case of Plane Stress

$E = 100$

$\nu = 0.0$

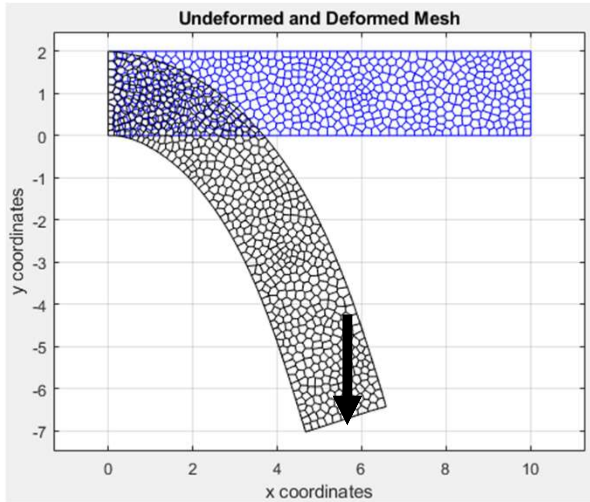
$L = 10$

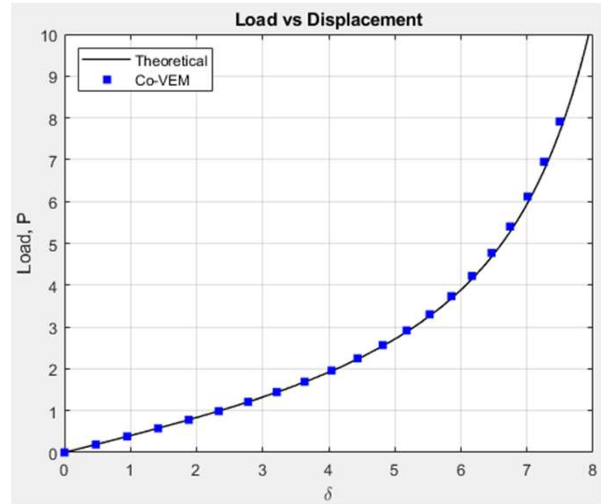
$t = 2$

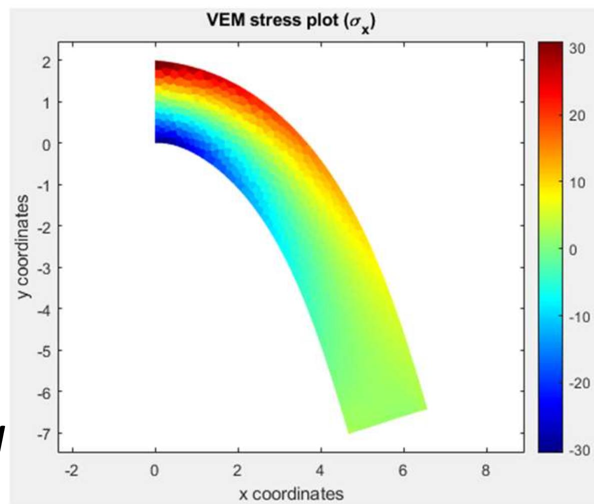
Number of V. Elements = 620

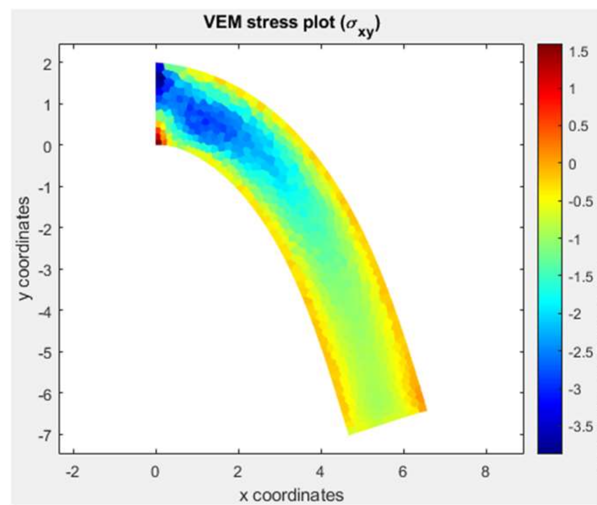
# of Arc Length Increments = 20

Stresses plotted in local co-rotated coordinates









Talischi et. al., "PolyMesher: …", *Struct and Multidiscip. Optimiz.*, vol. 6, issue 3, 2012.

6. Numerical Examples

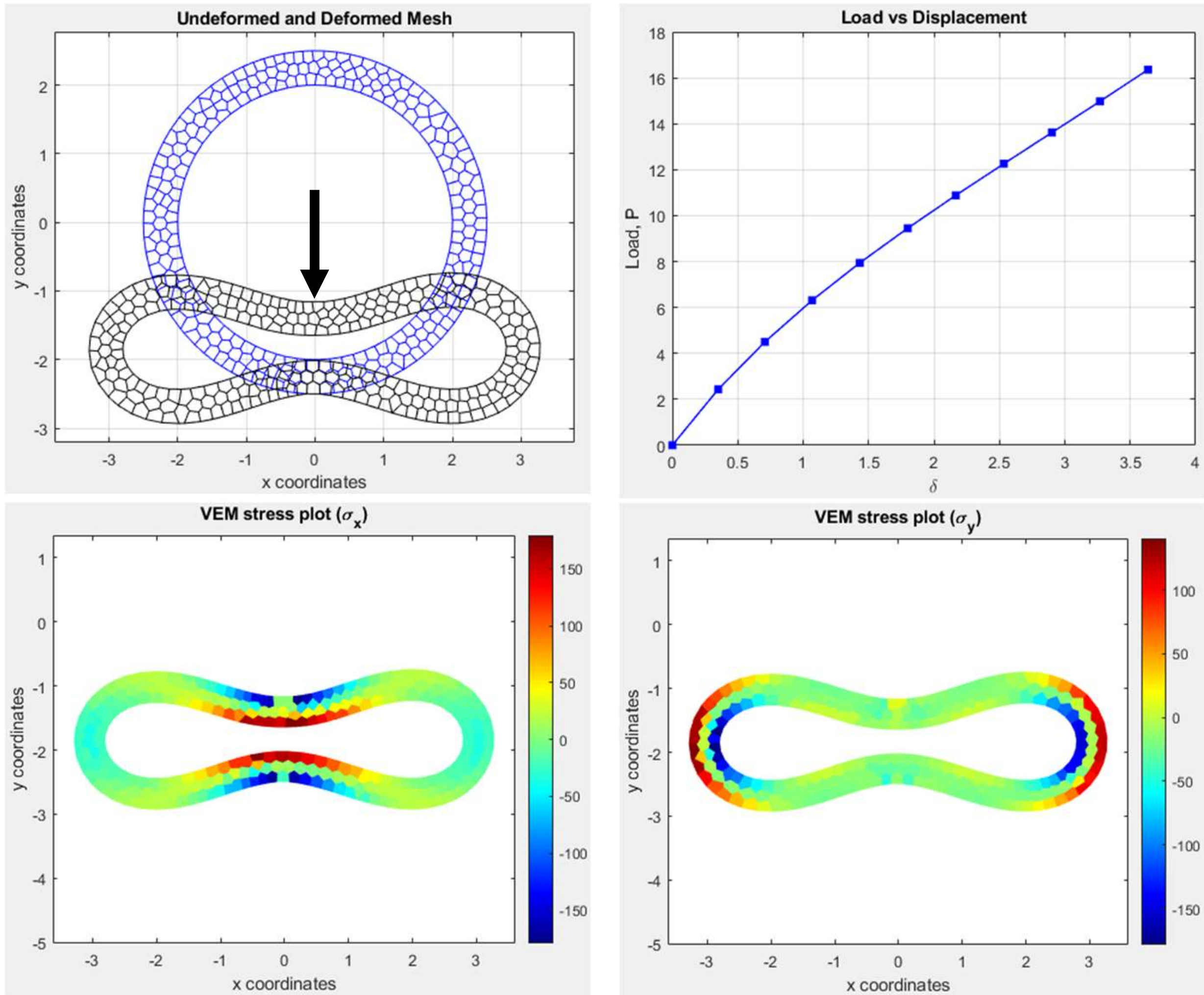
Linear Elastic Ring

Case of Plane Stress

$$E = 1000$$

$$\nu = 0.3$$

$$t = 1$$



Number of V. Elems = 250

# of Arc Len. Incrs = 10

Stresses plotted in local co-rotated coordinates

6. Numerical Examples

Linear Elastic Arch

Case of Plane Stress

$$E = 1000$$

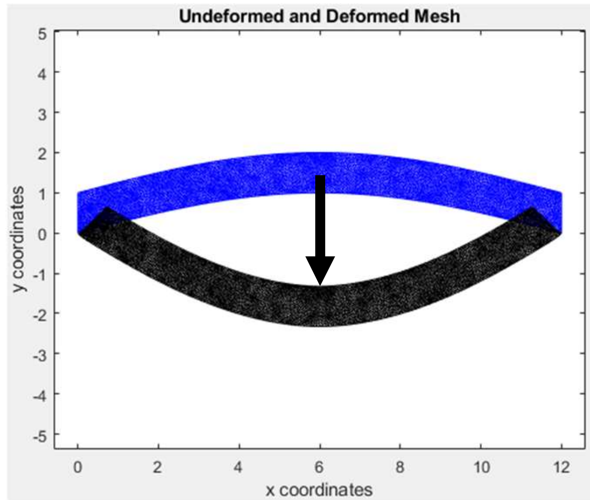
$$\nu = 0.3$$

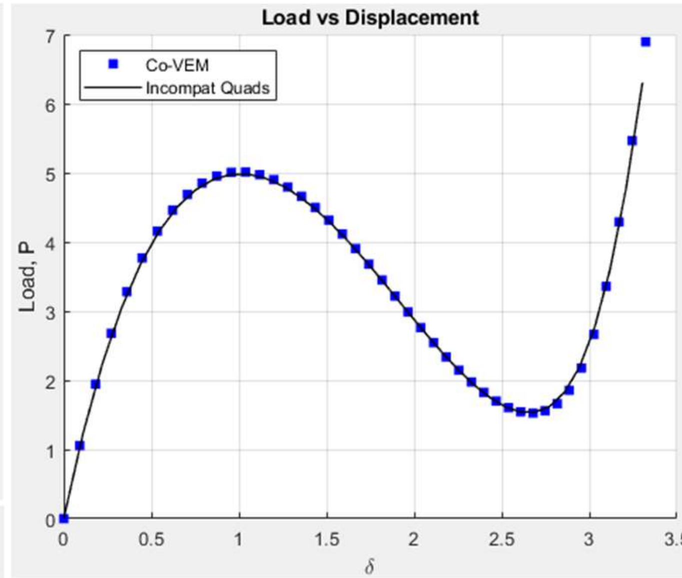
$$L = 12$$

$$t = 1$$

Number of V. Elements = 6000

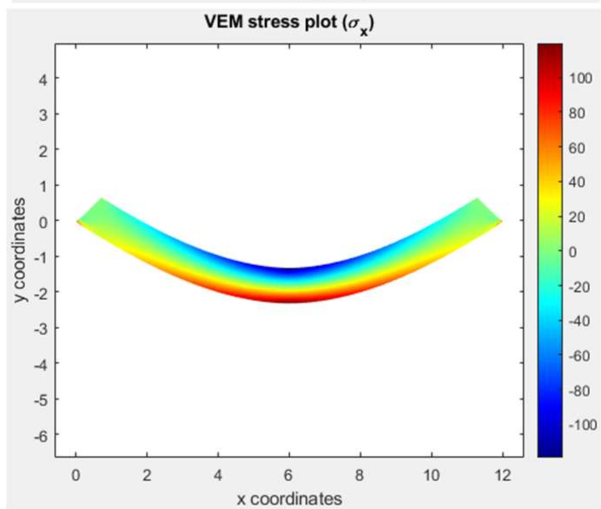
# of Arc Length Increments = 43





Results match co-rotational Incompatible modes quadrilateral element

Stresses plotted in local co-rotated coordinates



6. Numerical Examples

Example: Elasto-Plastic Cantilever

Case of Plane Stress Plasticity

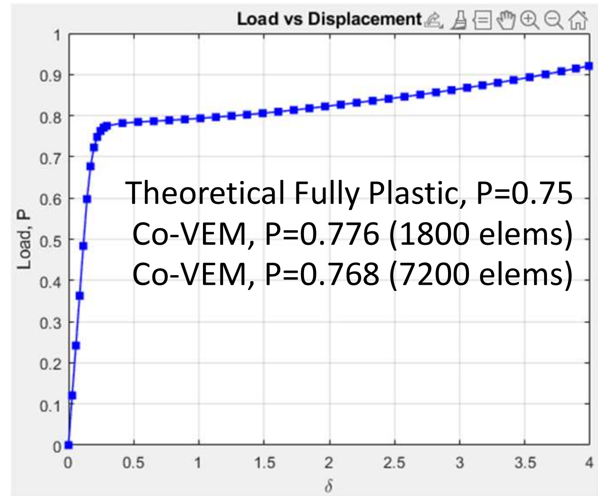
$$E = 29000, \; E_h = 1, \; \sigma_y = 36$$

$$\nu = 0.3$$

$$L = 12$$

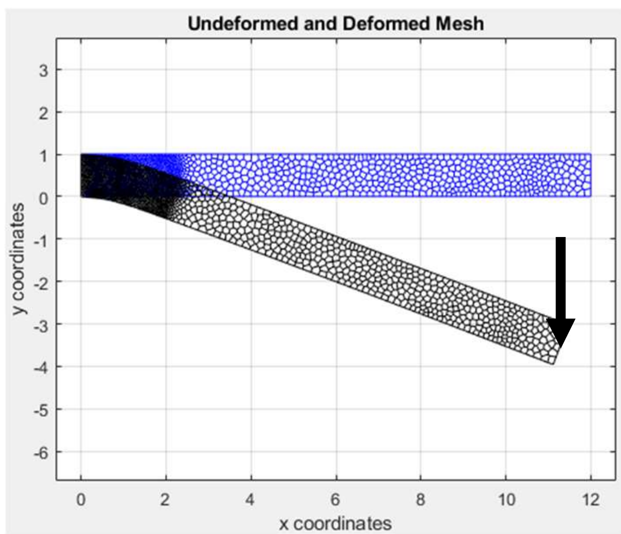
$$t = 1$$

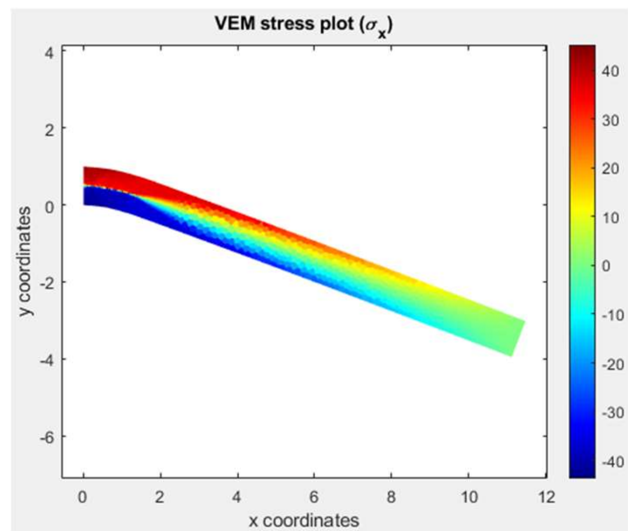
Number of Elements = 1800
(with element density biased toward support)



# of Arc Length
Increments = 42

Stresses plotted in local
co-rotated coordinates





6. Numerical Examples

Example: Elasto-Plastic Ring

Case of Plane Stress Plasticity

$$E = 1000,\ E_h = 1$$

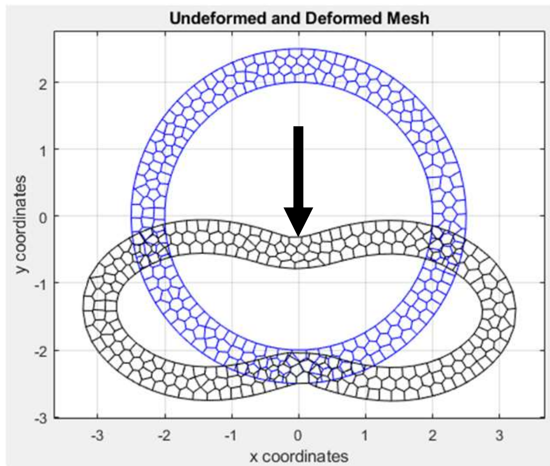
$$\sigma_y = 36$$

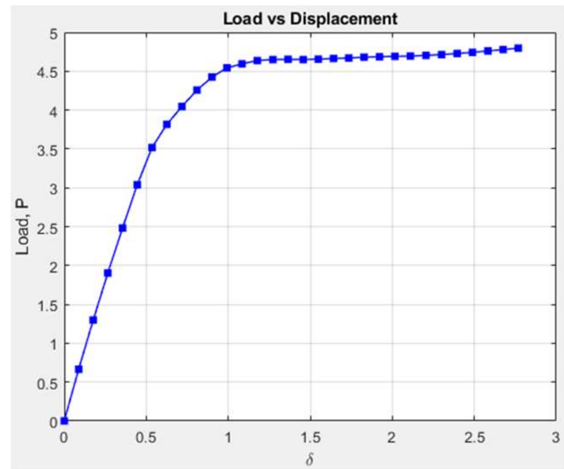
$$\nu = 0.3$$

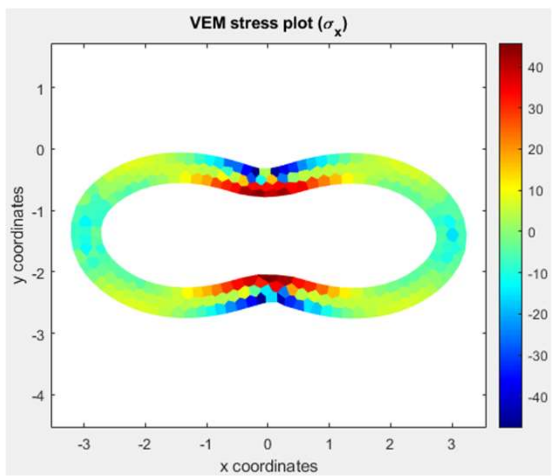
$$t = 1$$

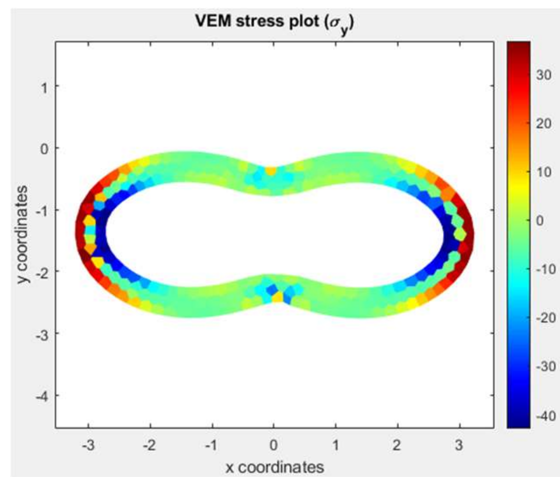
Number of Elements = 250

\# of Arc Length
Increments = 30



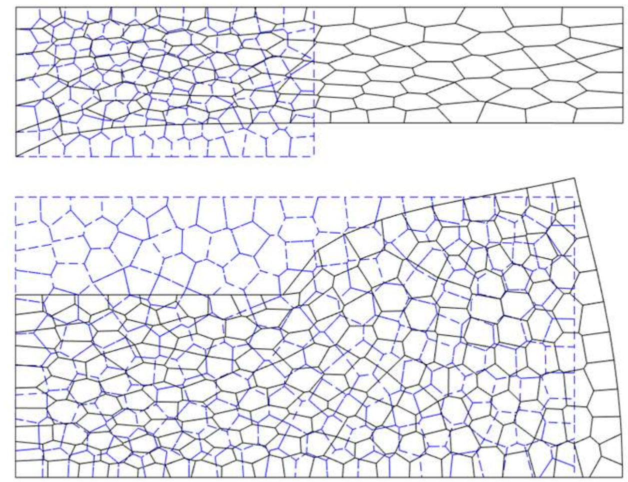






7. Conclusions

- Co-rotation is a viable method to include large displacements and rotations with VEM
- Feasible with elasticity and plasticity
- Verified with representative benchmark problems



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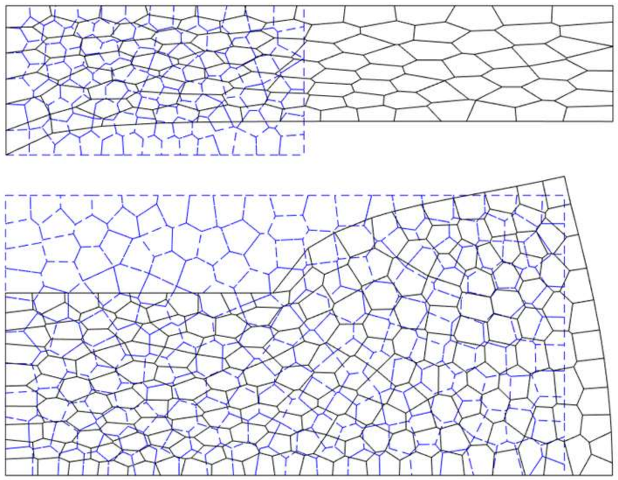

Possible Future work

- Finite Strains
- Investigate plasticity convergence with stabilization free VEM
- Explore consistent linearization with projectors

8. Questions

Questions?