Theoretical Effective Length Factor for Pinned-Fixed Column
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1 Introduction

The theoretical effective length factor, k, for a pinned-fixed column is often reported incorrectly as 0.707 or $1/\sqrt{2}$ in textbooks and across the internet. The correct value, rounded to four digits, is $k = 0.6992$. It is likely that this number has at some time been rounded to 0.7, then still others assume this comes from the common number $\sqrt{2}/2$ or cosine of 45 degrees. To set the record straight the fourth order differential equation of column buckling is derived. Pinned-fixed boundary conditions are imposed upon the solution to the differential equation. Finally, the effective length factor for a pinned-fixed column is found. Although this solution is correctly found in some textbooks [5], all the steps and details are not given. Hence, herein, it is shown that the correct theoretical value for the pinned-fixed case is $k = 0.6991556596428472\ldots$

2 Relationship between curvature and strain

Consider a beam undergoing a lateral displacement as shown in Figure 1. From this beam consider the differential element dx before and after deformation. The coordinate y is measured from the neutral axis of the beam. In the undeformed state the beam fiber at height y has length

$$ds = dx = \rho d\theta. \tag{1}$$

The above equation (1) results owing to the fact that in the undeformed state $dx = ds$ and in the deformed state the length of the differential element at the neutral axis is unchanged. As a result $dx = \rho d\theta$. After deformation the beam fiber at height y has length

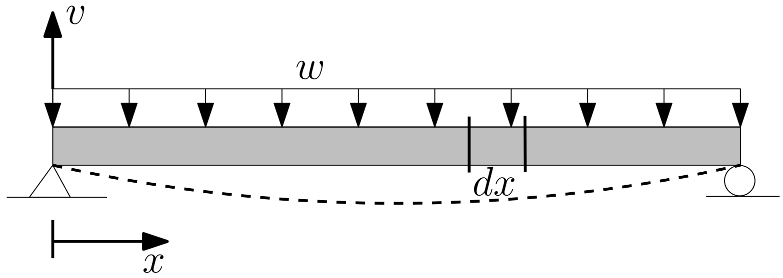
$$ds' = (\rho - y)d\theta. \tag{2}$$

For small displacements and rotations the strain is change in length over original length.

$$\varepsilon = \frac{ds' - ds}{ds} = \frac{(\rho - y)d\theta - \rho d\theta}{\rho d\theta} = \frac{-y d\theta}{\rho d\theta} = \frac{-y}{\rho} \tag{3}$$

From (3), curvature $1/\rho$, is related to strain as

$$\frac{1}{\rho} = \frac{-\varepsilon}{y} \tag{4}$$



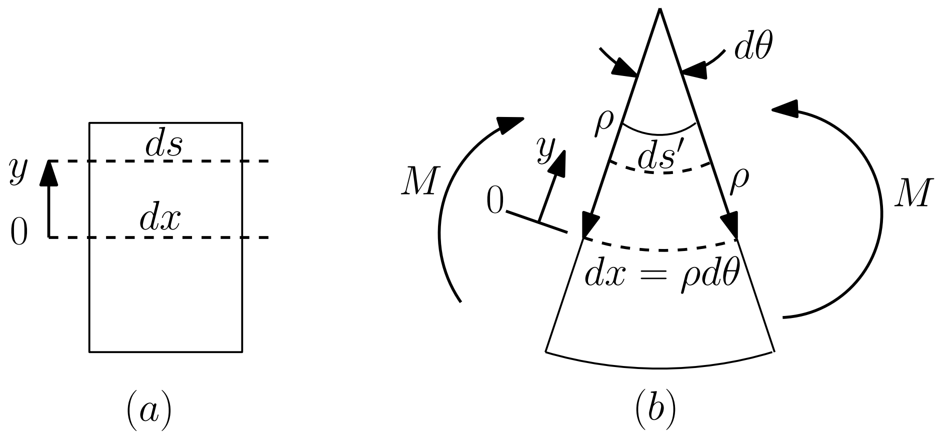


Figure 1: Lateral Deflection Curvature: (a) Undeformed differential element dx, (b) Deformed differential element dx.

3 Moment curvature relation

For a homogeneous linear elastic material with modulus of elasticity, E, stress, σ, is related to strain, ε, via Hooke's law.

$$\sigma = E\varepsilon \quad or \quad \varepsilon = \frac{\sigma}{E}. \tag{5}$$

Recall from basic mechanics of materials [3] that normal stress, σ, is related to moment with y coordinate measured positive upwards from the neutral axis of the beam.

$$\sigma = \frac{-My}{I} \tag{6}$$

Then combine (6) with (5) to get

$$\varepsilon = \frac{-My}{EI}. \tag{7}$$

Next substitute (7) into (4) to obtain

$$\frac{1}{\rho} = -\frac{-My}{yEI} = \frac{M}{EI}. \tag{8}$$

From a basic calculus textbook [4] curvature is shown to have the following formula

$$\frac{1}{\rho} = \frac{\frac{d^2v}{dx^2}}{[1 + (\frac{dv}{dx})^2]^{\frac{3}{2}}}, \tag{9}$$

where $v(x)$ is the equation of the curve for which the curvature is being calculated. For our beam, v, is the lateral displacement of the beam. Now, for small displacements and small rotations $\frac{dv}{dx} \approx 0$ so that (9) becomes

$$\frac{1}{\rho} = \frac{d^2v}{dx^2}. \tag{10}$$

Finally, substituting (10) into (8) yields

$$\frac{M}{EI} = \frac{d^2v}{dx^2}. \tag{11}$$

4 Derivation of fourth order differential equation of buckling

Often column buckling problems are solved by deriving and solving a second order differential equation. However, the fourth order differential equation of buckling is more intuitive when trying to understand how to apply boundary conditions. In particular, the boundary conditions for the pinned-fixed case are easier to understand by using the fourth order differential equation.

To begin consider the case of a column with axial load, P, distributed load, $w(x)$, and lateral deflection, $v(x)$, as shown in Figure 2. A differential element dx is cut from the deformed shape of the column. Equilibrium equations are written for the differential element as follows:

$$+\uparrow \sum F_y = 0; \quad V - wdx - (V + dV) = 0 \quad \Rightarrow \quad dV = -wdx \quad \Rightarrow \quad \frac{dV}{dx} = -w. \tag{12}$$

$$+\circlearrowleft \sum M_o = 0; Pdv - Vdx + \frac{wdx^2}{2} - M + M + dM = 0 \quad \Rightarrow \quad Pdv - Vdx + dM = 0, \tag{13}$$

where the last expression in (13) arises since higher order terms, dx^2, go to zero faster than other terms in the limit as $dx \to 0$.

Next use (13) and divide all terms by dx to get

$$P\frac{dv}{dx} + \frac{dM}{dx} = V. \tag{14}$$

Then use (11) in (14) so that

$$P\frac{dv}{dx} + \frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) = V. \tag{15}$$

For constant column material and cross-section (15) becomes

$$P\frac{dv}{dx} + EI\frac{d^3v}{dx^3} = V. \tag{16}$$

Substituting (16) into (12)

$$P\frac{d^2v}{dx^2} + EI\frac{d^4v}{dx^4} = -w. \tag{17}$$

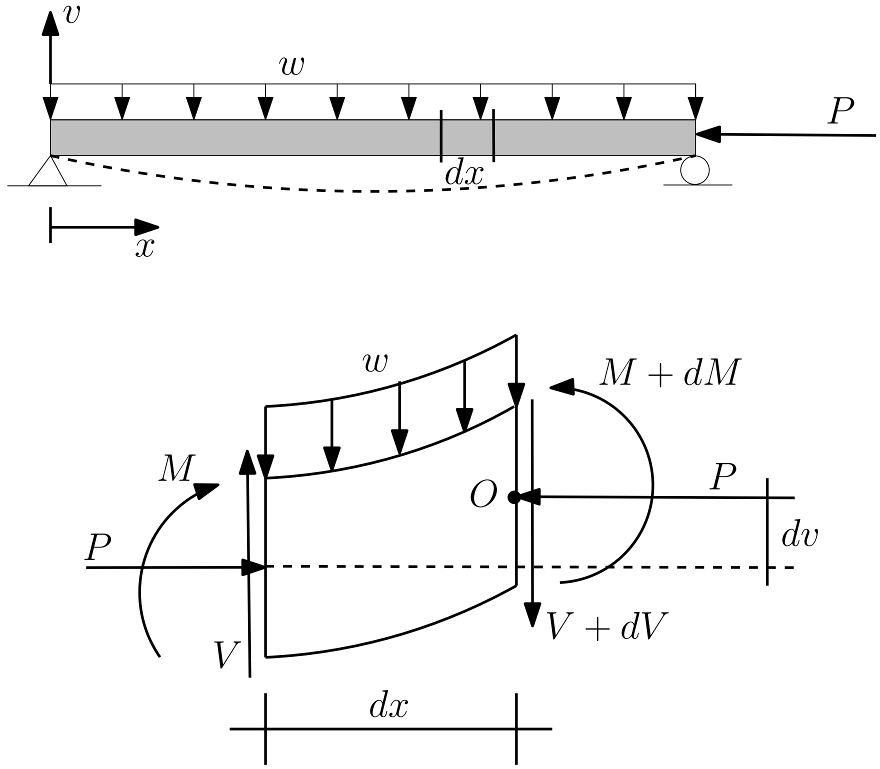


Figure 2: Freebody Diagram of Differential Element

Dividing through with EI and rearranging yields

$$\frac{d^4v}{dx^4} + \frac{P}{EI}\frac{d^2v}{dx^2} = \frac{-w}{EI}. \tag{18}$$

It is possible to consider cases of buckling that include the effect of a nonzero w applied laterally to the column. However, for the purposes herein it is appropriate to assume $w = 0$. As a result,

$$\frac{d^4v}{dx^4} + \frac{P}{EI}\frac{d^2v}{dx^2} = 0. \tag{19}$$

Finally, for convenience, setting $\lambda^2 = \frac{P}{EI}$ and substituting into (19) yields

$$\frac{d^4v}{dx^4} + \lambda^2\frac{d^2v}{dx^2} = 0. \tag{20}$$

Equation (20) has the general solution

$$v = A\sin\lambda x + B\cos\lambda x + Cx + D. \tag{21}$$

5 Buckling of a pinned-fixed column

The fourth order differential equation (20) and the accompanying solution (21) are a firm basis for determining the theoretical buckling load of columns with various boundary conditions. In particular, the case of a pinned-fixed column, Figure 3, has the following boundary

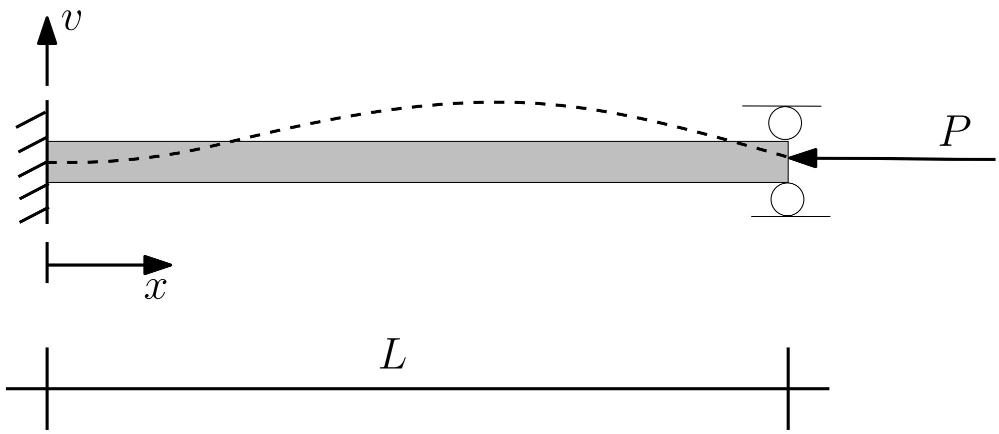


Figure 3: Pinned-Fixed Column With Buckled Shape Shown

conditions and corresponding equations using the general solution (21):

$$\begin{aligned}
&At\ x=0,\ v=0 \quad \Rightarrow \quad B+D=0\\
&At\ x=0,\ \frac{dv}{dx}=0 \quad \Rightarrow \quad A\lambda+C=0\\
&At\ x=L,\ v=0 \quad \Rightarrow \quad A\sin\lambda L+B\cos\lambda L+CL+D=0\\
&At\ x=L,\ \frac{d^2v}{dx^2}=0 \quad \Rightarrow \quad -A\lambda^2\sin\lambda L-B\lambda^2\cos\lambda L=0.
\end{aligned} \tag{22}$$

The four equations in (22) may be written in matrix form

$$\begin{bmatrix} 0 & 1 & 0 & 1\\ \lambda & 0 & 1 & 0\\ \sin\lambda L & \cos\lambda L & L & 1\\ -\lambda^2\sin\lambda L & -\lambda^2\cos\lambda L & 0 & 0 \end{bmatrix} \begin{Bmatrix} A\\ B\\ C\\ D \end{Bmatrix} = \begin{Bmatrix} 0\\ 0\\ 0\\ 0 \end{Bmatrix}. \tag{23}$$

The trivial solution of (23) is $A=B=C=D=0$, which is the uninteresting case of zero lateral displacement and hence no buckling. The nontrivial solution is for the determinant of the coefficient matrix to equal zero. In order to take the determinant of a 4x4 matrix a cofactor expansion [1] is required, in this case about the first row. Recall that these operations are multiplied by relevant signs as shown in the following matrix:

$$\begin{bmatrix} 1 & -1 & 1 & -1\\ -1 & 1 & -1 & 1\\ 1 & -1 & 1 & -1\\ -1 & 1 & -1 & 1 \end{bmatrix}. \tag{24}$$

Hence, taking the determinant of the coefficient matrix about the first row yields

$$\begin{aligned}
&0+(-1)(-\lambda^2\sin\lambda L)+0+(-1)((-\lambda^2\cos\lambda L\sin\lambda L+\lambda^3L\cos\lambda L+\lambda^2\cos\lambda L\sin\lambda L)=0\\
&\Rightarrow \lambda^2\sin\lambda L+\lambda^2\cos\lambda L\sin\lambda L-\lambda^3L\cos\lambda L-\lambda^2\cos\lambda L\sin\lambda L=0\\
&\Rightarrow \sin\lambda L+\cos\lambda L\sin\lambda L-\lambda L\cos\lambda L-\cos\lambda L\sin\lambda L=0\\
&\Rightarrow \tan\lambda L+\sin\lambda L-\lambda L-\sin\lambda L=0\\
&\Rightarrow \tan\lambda L=\lambda L.
\end{aligned} \tag{25}$$

The final expression of (25) is a transcendental equation in terms of λL. The lowest nonzero value that satisfies (25) is

$$\lambda L = 4.4934094579090\ldots \tag{26}$$

Further, note that

$$\lambda = \frac{4.4934094579090}{L} \tag{27}$$

Now recall that

$$\frac{P}{EI} = \lambda^2 = \left(\frac{4.4934094579090}{L}\right)^2. \tag{28}$$

Upon rearranging (28) it is found that

$$P = \frac{EI(4.4934094579090)^2}{L^2}. \tag{29}$$

To find the effective length factor, k, set (29) equal to the standard form for Euler critical buckling load and solve for k, that is

$$P_{cr} = \frac{\pi^2 EI}{(kL)^2} = \frac{EI(4.4934094579090)^2}{L^2}. \tag{30}$$

After some algebra solving for k it is found that

$$k = \frac{\pi}{4.4934094579090} = 0.6991556596428. \tag{31}$$

The above value is consistent with the value provided in the Theory of Elastic Stability by Timoshenko and Gere [5]. As a result the correct critical buckling load and effective length factor for the pinned fixed case is

$$P_{cr} = \frac{\pi^2 EI}{(0.6992L)^2} \quad and \quad k = 0.6991556596428. \tag{32}$$

6 Conclusion

A derivation of the theoretical critical buckling load and effective length factor for a pinned fixed column is provided. It is shown that the effective length factor is $k = 0.699\ldots$ rather than 0.707 or $1/\sqrt{2}$, which is often erroneously printed in textbooks or on the internet. Furthermore, in the process the fourth order differential equation of buckling for columns was found. This differential equation is very useful for examining all of the other standard column buckling cases, such as pinned pinned, fixed fixed, fixed free, fixed pinned with sway, and fixed fixed with sway, as well. These other standard theoretical cases are presented as a useful reference in Figure 4, which includes information adapted from [2].

References

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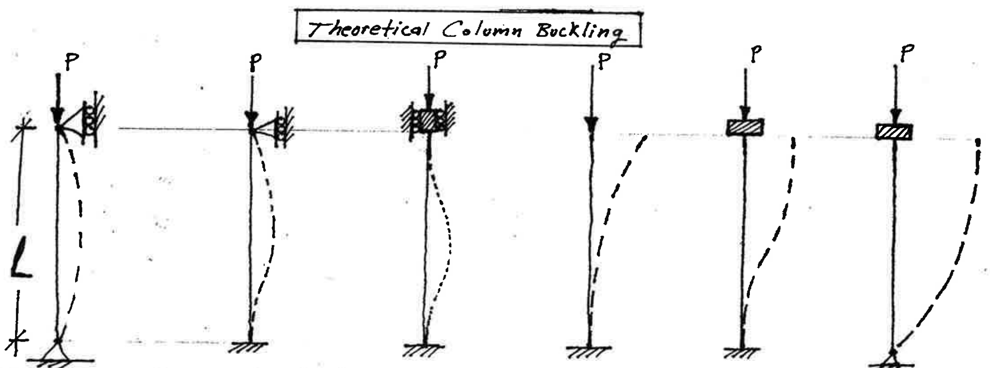

Theoretical Effective Length Factor K	1.0	0.69916	0.5	2.0	1.0	2.0
Number of $\frac{1}{2}$ sine waves $n = \frac{1}{K}$	1.0	1.4303	2.0	0.5	1.0	0.5
P_{cr} Euler Buckling Capacity	$\frac{\pi^2 EI}{L^2}$	$\frac{\pi^2 EI}{(0.6992L)^2}$	$\frac{\pi^2 EI}{(0.5L)^2}$	$\frac{\pi^2 EI}{(2L)^2}$	$\frac{\pi^2 EI}{L^2}$	$\frac{\pi^2 EI}{(2L)^2}$
$\frac{P_{cr}}{P_{cr\,(\text{pinned pinned})}}$	1.0	2.046	4.0	0.25	1.0	0.25
Recommended Design Values When ideal Conditions are approximated K	1.0	0.8	0.65	2.1	1.2	2.0

Euler Buckling Formula, $P_{cr} = \frac{\pi^2 EI}{(KL)^2}$

Figure 4: Theoretical Column Buckling Information for Various Boundary Conditions