Aspects of Static Nonlinear Finite Element Analysis for Solids and Structures

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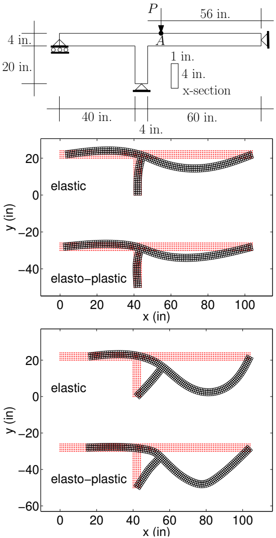
Outline

1. What is Finite Element Analysis?
2. Linear vs nonlinear analysis
3. Types of nonlinearity
4. Techniques for following nonlinear path
5. Why solve such problems?
6. Math involved
7. Conclusion
8. Questions



What is Finite Element Analysis?

Why do it? What does it do for us?

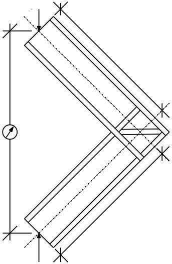
3D Field Variables

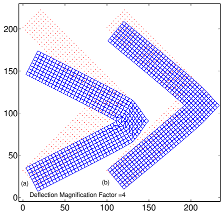
- displacements

  $u_x(x, y, z)$

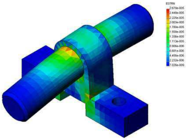
  $u_y(x, y, z)$

  $u_z(x, y, z)$



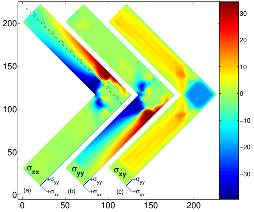


- 6 independent strains $\varepsilon_{xx}(x, y, z), \ldots \varepsilon_{zx}(x, y, z)$



- 6 independent stresses

  $\sigma_{xx}(x, y, z), \ldots \sigma_{zx}(x, y, z)$



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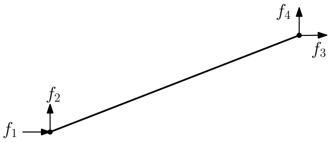
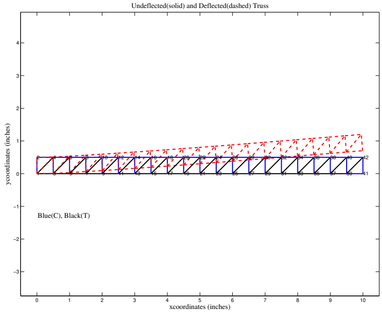
What is Finite Element Analysis?

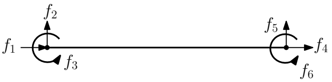


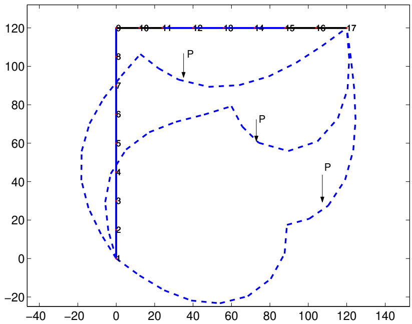


- Define Structure with "finite elements"
- Truss Elements
- axial
- line shaped

What is Finite Element Analysis?

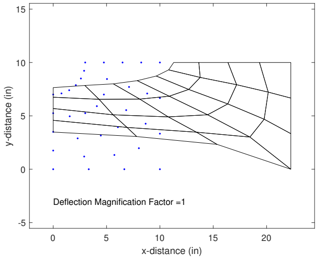
- Frame Elements
- axial, shear, moment
- line shaped

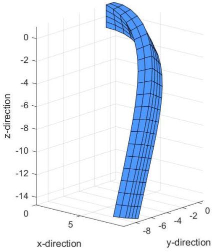




What is Finite Element Analysis?

- Solid Elements
- normal stresses and shear stresses
- 2D area (triangles, quadrilaterals)
- 3D volume (tetrahedral, cubic)





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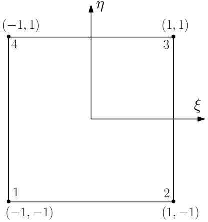
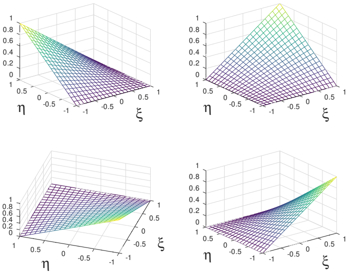
What is Finite Element Analysis?

- Define shape functions for element
- Interpolate nodal displacement values within element
- Example: Quadrilateral shape function at node r in (ξ, η) parent coordinates

$$N_r = \frac{1}{4}(1 + \xi_r \xi)(1 + \eta_r \eta)$$

$$u_x = \sum_{r=1}^{n=4} N_r u_x^r, \quad u_y = \sum_{r=1}^{n=4} N_r u_y^r$$



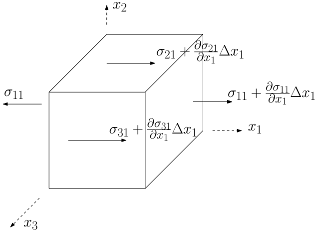


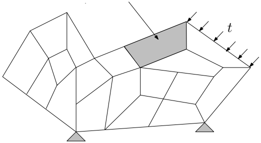
What is Finite Element Analysis?

- Differential Volume Equilibrium

  $\sigma_{ji,j} + b_i = \rho a_i$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + b_1 = 0$$
$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + b_2 = 0$$
$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + b_3 = 0$$





- Constitutive relations $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$
- Compatibility - for FEA enforced by inter-element connectivity

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What is Finite Element Analysis?

Some relevant formulas

- Find strain displacement relations $\varepsilon = Bd$
- Use fundamental lemma of calculus of variations
- $\Rightarrow \int_\Omega w_i(\sigma_{ij,j} + b_i)d\Omega = 0$
- Construct the local element matrices, $k_e u = f_e$, where $k_e = \int_\Omega B^T CBd\Omega$, $f_e = \int_\Gamma w_i h_i d\Gamma + \int_\Omega w_i b_i d\Omega$
- Assemble the local element stiffness matrix and local element force vector into global stiffness and force equations
- $KU = F$, where $K = \overset{n_{el}}{\underset{e=1}{A}} k_e$, $F = \overset{n_{el}}{\underset{e=1}{A}} f_e$, and A is the assembly operator.

What is linear finite element analysis?

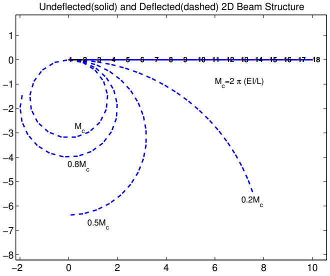
- $KU = F$
- K is independent of displacements
- Put in load F and get out displacement $U = (K^{-1}F)$
- Can be done in a single load step
- Plot of load versus displacement is linear
- Easy and computational cost is low

What is linear Finite Element Analysis?

A Rough Overview

- Choose element
- Mesh the domain
- Find strain displacement relations
- Put strains into integral form of equilibrium equations
- Construct local stiffness and load vector for each element
- Assemble global stiffness and global forces
- Solve global equations for global displacements
- Post processing, find stresses and strains from global displacements

When is linear finite element analysis no longer valid?

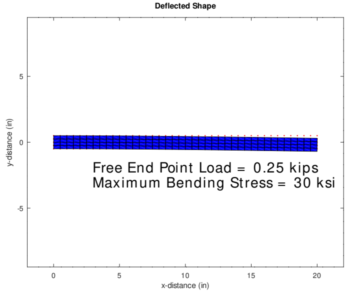


- Large displacements or large rotations
- Hyperelastic material loaded beyond linear range
- Material is loaded beyond the yield point
- Contact problems

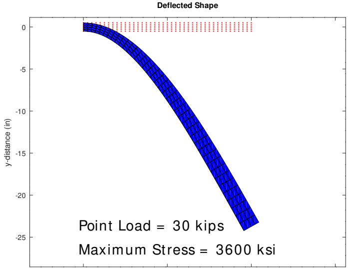
When is linear finite element analysis no longer valid?

An example of linear program used beyond a reasonable level

- Linear Analysis, small displacements, $\sigma_{max} < \sigma_{Yield}$



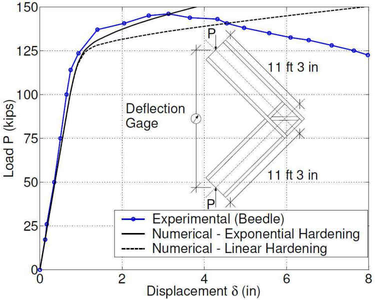
- Linear Analysis, large displacements erroneous results, $\sigma_{max} > \sigma_{Yield}$

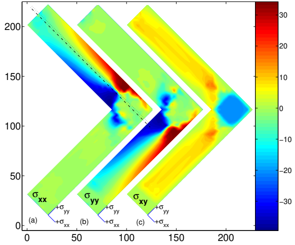


What is nonlinear finite element analysis?

- Relationship between load and displacement is not linear
- Types of nonlinearity
  - Material nonlinearity (inelasticity, hyperelasticity)
  - Geometric nonlinearity
  - Material and Geometric nonlinearity
  - Change in supports or contact

Nonlinear Examples - Material nonlinearity, Plasticity





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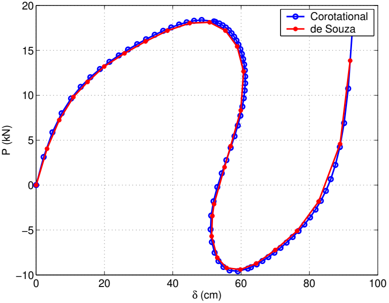
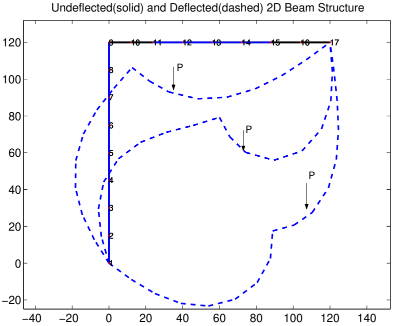
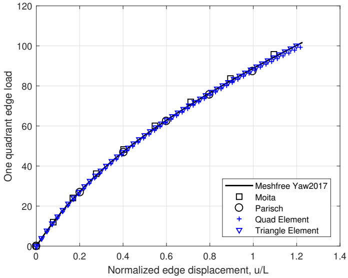
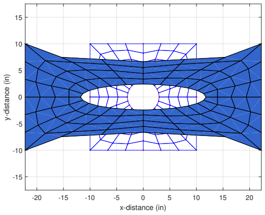
Nonlinear Examples - Geometric nonlinearity





Geometric and material(hyperelasticity) nonlinearity





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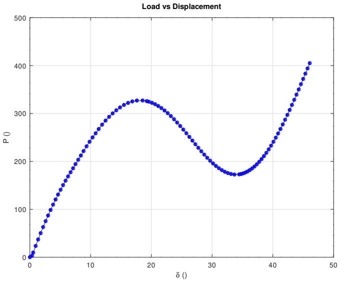
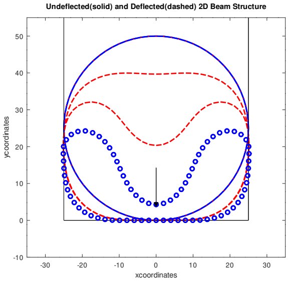
Nonlinearity due to change in support or contact (and geometric nonlinearity)





Why solve nonlinear problems?

- Some Example Reasons
  - To characterize failure, brittle or ductile
  - Post yield and or buckling behavior as in a crash test
  - Energy absorption capability through nonlinear regime
  - Buckling imperfection sensitivity analysis
  - Verifying structural collapse rather than numerical collapse
  - Metal Forming Analysis (springback)
  - To determine expected large displacements

Ingredients for nonlinear finite element analysis

- Strain displacement relations
- Appropriate Material Model
- Formulation of the incremental equations for nonlinear analysis
- Equilibrium path following scheme
- Iterative scheme to maintain equilibrium within each load step

Examples of the various ingredients for nonlinear FEA

- Strain displacement relations (Green strain tensor)

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right)$$

- Strain displacement relations using 2D quadrilateral shape functions

$$E = Bu = B_L u + B_{NL} u, \quad E = \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{12} \end{Bmatrix}$$

Examples of the various ingredients for nonlinear FEA

- Displacement, Stretch, or Strain dependent displacement hyperlastic materials
- 2D incompressible Ogden Material Model (strain energy density function) in terms of stretches (λ_1, λ_2)

$$W(\lambda_1, \lambda_2) = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_1^{-\alpha_p}\lambda_2^{-\alpha_p} - 3)$$

- From strain energy density function determine principle stresses $(\lambda_i^2 = 2E_i + 1)$

$$\sigma_i = \frac{\partial W}{\partial E_i} = \frac{\partial W}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial E_i}$$

- Principle stress principle strain relations $(\Delta\sigma_i = C_{ij}\Delta E_j)$

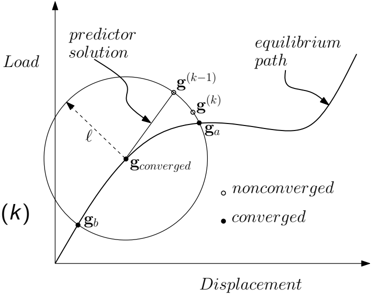
$$C_{ij} = \frac{\partial^2 W}{\partial E_i \partial E_j}$$

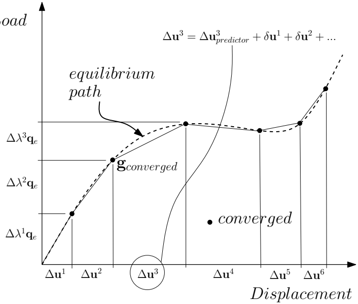
- Many incremental steps required

$$K_n^{t(k)}\Delta U_n^{(k)} = F_{n+1}^{ext} - F_n^{int(k)} = g^{(k)}$$

- (k) is the Newton-Raphson iteration counter within a given increment

$$\Delta U_n^{(k+1)} = U_n^{(k)} + \delta U^{(k+1)}$$
$$\delta U^{(k+1)} = -(K_n^{t(k)})^{-1} g^{(k)}$$





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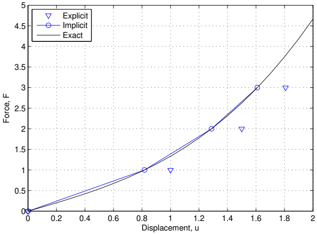
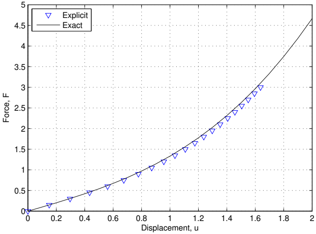
Incremental Equations w/o equilibrium iterations

- Solution drifts from equilibrium path
- Iterations bring solution back to equilibrium
- More increments required w/o iterations





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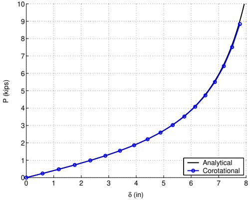
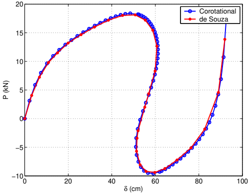
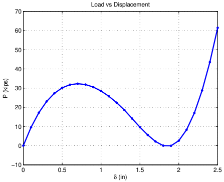
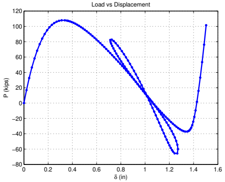
Techniques for following nonlinear path

- Load Control
- Displacement Control (Snap through)
- Arc Length Control (Snap through and Snap back)
- Arc Length Control (Multiple Windings)

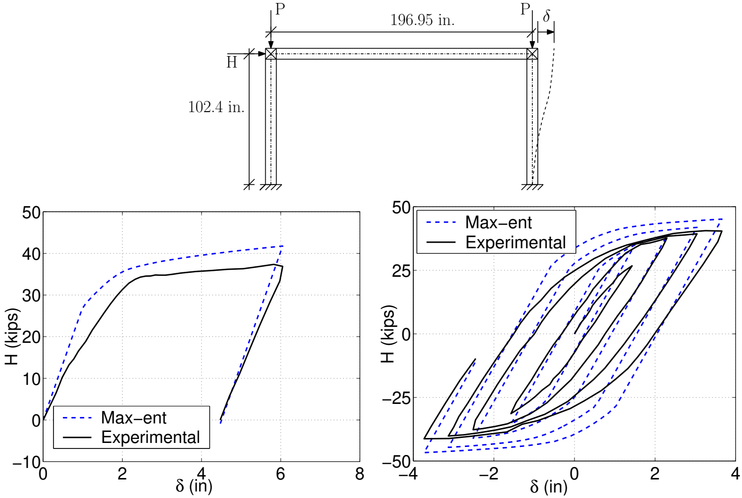








Portal frames - monotonic and cyclic loading

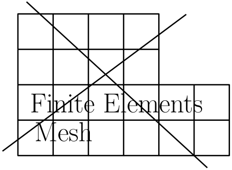


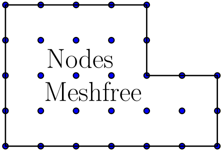
(Experimental – Toma et al. J. Constr. Steel Rsrch. 1995)

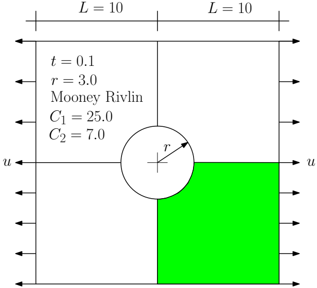
The math involved

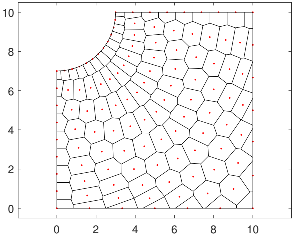
- Calculus
- Tensors
- Partial Differential Equations
- Extensive Linear Algebra (inverse, eigenvalues, matrix algebra)
- Numerical Integration
- Computational Geometry
- Incremental iterative Newton-Raphson path following schemes

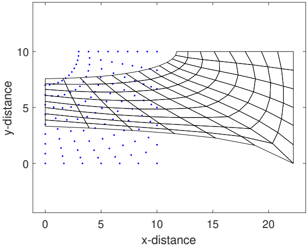
Alternatives to finite element analysis - Meshfree











Conclusions

- Theory
- Experiments
- Computational Methods (here to stay)
- Nonlinear FEA analysis requires experience and correct modeling of material behavior
- Many other applications besides solid mechanics and structures (fluid mechanics, magnetic fields, acoustics, heat transfer, etc.)

Questions

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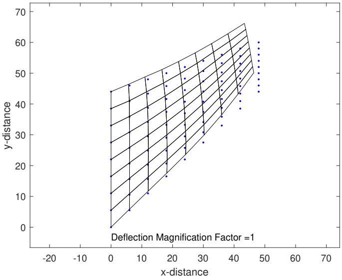
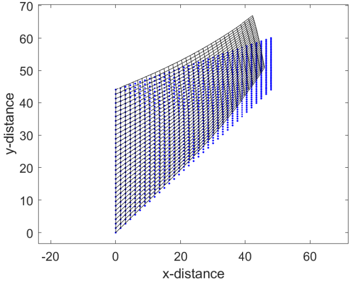
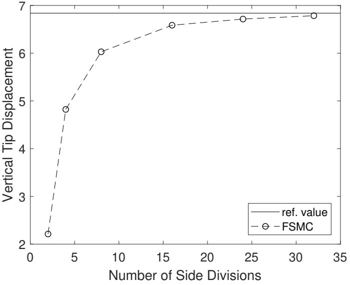
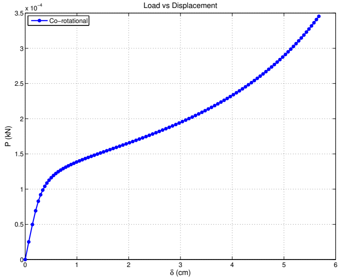
Mesh Refinement

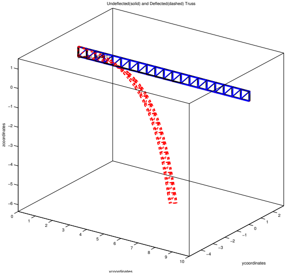




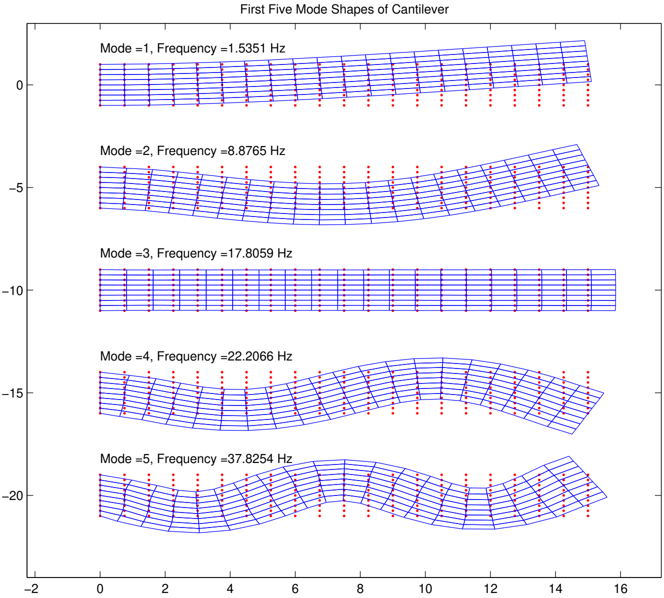


Lateral Torsional Buckling





Vibration Mode Shapes and Frequencies



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Stresses Represented with Color Chart