3D Co-rotational Truss Formulation

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November 28, 2011

key words: geometrically nonlinear analysis, 3D co-rotational truss, corotational truss, variationally consistent, load control, displacement control, generalized displacement control, examples, algorithms, Newton-Raphson, iterations

1 Introduction

This article presents information necessary for a three-dimensional co-rotational truss formulation. The truss structure is allowed to have arbitrarily large displacements and rotations at the global level (so long as local truss element strains are small the results are valid). All truss elements are assumed to remain linear elastic. As with any co-rotational formulation three ingredients are required. They are (i) the relations between global and local variables, (ii) the angle(s) of rotation of a co-rotating frame, and (iii) a variationally consistent tangent stiffness matrix. The approach closely follows that provided by Crisfield [2]. Each of the ingredients are presented below, however, first some preliminary information is developed.

2 Co-rotational Concept

Let us consider the co-rotational concept in terms of truss elements. As a truss structure is loaded the entire truss deforms from its original configuration. During this process an individual element potentially does three things; it rotates, translates and deforms. The global displacements of the end nodes of the truss element include information about how the truss element has rotated, translated and deformed. The rotation and translation are rigid body motions, which may be removed from the overall motion of the truss. If this is done, all that remains are the strain causing deformations of the truss element. The strain causing local deformations are related to the force induced in the truss element. A co-rotational formulation seeks to separate rigid body motions from strain producing deformations at the local element level. This is accomplished by attaching a local element reference frame (or coordinate system), which rotates and translates with the truss element. For 3D trusses this amounts to attaching a co-rotating coordinate frame to the truss such that the x-axis is always directed along the truss element (see Figure 1). With respect to this local co-rotating coordinate frame the rigid body rotations and translations are zero and only local strain producing deformations along the x-axis remain.

3 Preliminary Information

Consider a typical truss member in its initial and current configurations as shown in Figure 1. For the truss member in its initial configuration the global nodal coordinates are defined as

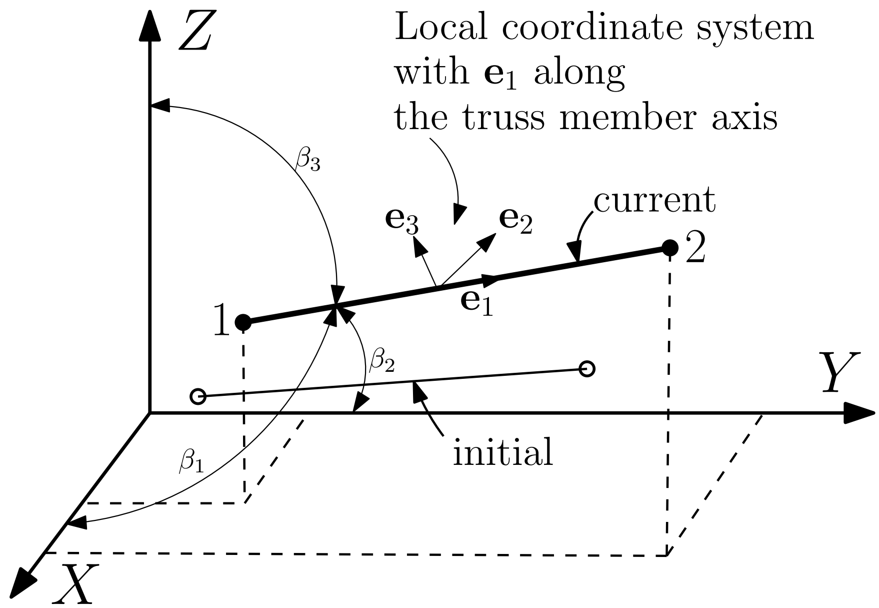


Figure 1: Initial and current configuration for typical truss member

(X_1, Y_1, Z_1) for node 1 and (X_2, Y_2, Z_2) for node 2. The original length of the truss member is

$$L_o = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}. \tag{3.1}$$

For the truss member in its current configuration the global nodal coordinates are $(X_1 + u_1, Y_1 + v_1, Z_1 + w_1)$ for node 1 and $(X_2 + u_2, Y_2 + v_2, Z_2 + w_2)$ for node 2, where for example, u_1, v_1, w_1 are the global nodal displacements of node 1 in the X, Y, Z directions respectively. The current length of the truss member is

$$L = \sqrt{L_1^2 + L_2^2 + L_3^2}, \tag{3.2}$$

where $L_1 = (X_2 + u_2) - (X_1 + u_1)$, $L_2 = (Y_2 + v_2) - (Y_1 + v_1)$ and $L_3 = (Z_2 + w_2) - (Z_1 + w_1)$. Note that L_1, L_2 and L_3 are simply the current distances between nodes 1 and 2 of the truss member in the global X, Y, Z directions respectively.

4 Relation between global and local deformations

Relations between global and local deformations are necessary to calculate the local axial deformation for a truss member. The axial deformation is used to calculate the internal force of the truss member in local coordinates. The axial deformation of the truss member, u_ℓ, is calculated as

$$u_\ell = L - L_o = \frac{L^2 - L_o^2}{L + L_o}, \tag{4.1}$$

where the last part is computationally better conditioned since the difference between L and L_o is possibly small. Note that the L and L_o are calculated in terms of global variables (the initial global coordinates and the current global nodal displacements). In the case of a truss element these relations, that allow extraction of the local deformation, are quite simple. However, in the case of beams, shells and continua, the relations are not necessarily so simple. In addition to the above relations it is also necessary to find relations between local and global axial forces. Such relations come about naturally in the process of finding the consistent tangent stiffness matrix and hence are left until that section.

5 Angles of rotation of the co-rotating frame

The global coordinates remain fixed throughout the co-rotational formulation. However, a local co-rotating coordinate frame is attached to each truss member as shown in Figure 1. This co-rotating coordinate frame rotates with the truss member as the truss structure deforms. The x axis of the local coordinate system is always defined along the current orientation of the truss member. This local x axis is described by the coordinate direction angles of the truss in its current configuration with respect to the global axes. The coordinate direction angles β_1, β_2 and β_3, of the truss in the current configuration are defined by the following relations

$$\cos\beta_1 = \frac{L_1}{L}, \quad \cos\beta_2 = \frac{L_2}{L}, \quad \cos\beta_3 = \frac{L_3}{L}. \tag{5.1}$$

6 Variationally consistent tangent stiffness matrix

To find the variationally consistent tangent stiffness matrix it is necessary to find 1) variational relations between local and global displacements and 2) relations between local and global axial forces. These two items provide the ingredients necessary to derive the variationally consistent tangent stiffness matrix.

6.1 Variational relations between local and global displacements

Consider a small movement $\delta\mathbf{d}_{21} = \delta\mathbf{d}_2 - \delta\mathbf{d}_1$ from the current configuration shown in Figure 2, where

$$\mathbf{d}_1 = \left\{ \begin{array}{c} u_1 \\ v_1 \\ w_1 \end{array} \right\} \quad \mathbf{d}_2 = \left\{ \begin{array}{c} u_2 \\ v_2 \\ w_2 \end{array} \right\}. \tag{6.1}$$

Using the dot product gives the components of $\delta\mathbf{d}_{21}$ along the direction $\mathbf{e}_1$. That is

$$\delta u_\ell = \mathbf{e}_1^T \delta\mathbf{d}_{21} = \left\{ \begin{array}{c} \cos\beta_1 \\ \cos\beta_2 \\ \cos\beta_3 \end{array} \right\}^T \delta\mathbf{d}_{21} \tag{6.2}$$

where $\mathbf{e}_1$ is the unit vector lying along the line drawn between the truss nodes in the current configuration. In subsequent writing $c_1 = \cos\beta_1$, $c_2 = \cos\beta_2$, $c_3 = \cos\beta_3$.

Equation (6.2) may be reexpressed as follows:

$$\delta u_\ell = \left\{ \begin{array}{c} \cos\beta_1 \\ \cos\beta_2 \\ \cos\beta_3 \end{array} \right\}^T \left\{ \begin{array}{c} \delta u_2 - \delta u_1 \\ \delta v_2 - \delta v_1 \\ \delta w_2 - \delta w_1 \end{array} \right\} = -c_1\delta u_1 + c_1\delta u_2 - c_2\delta v_1 + c_2\delta v_2 - c_3\delta w_1 + c_3\delta w_2 \tag{6.3}$$

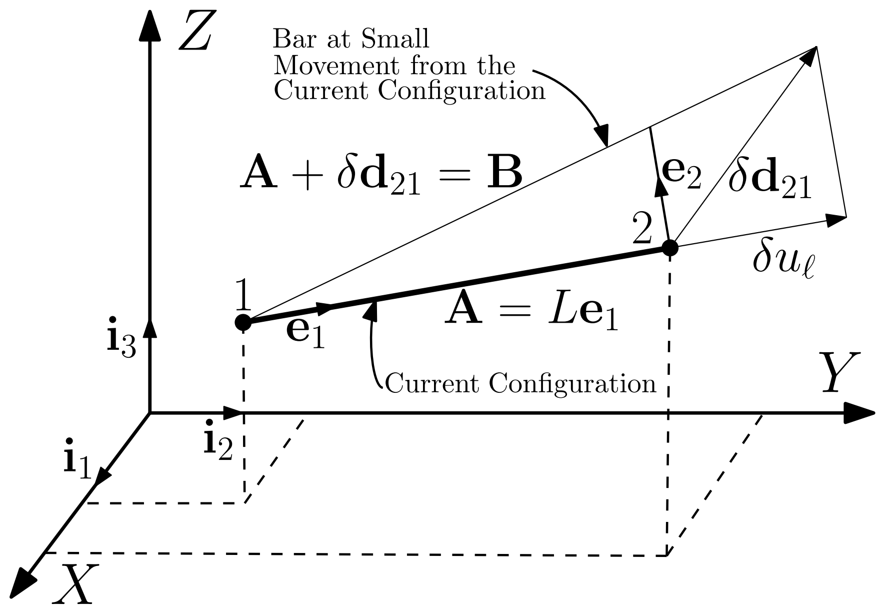


Figure 2: Small movement from current configuration.

$$= -c_1\delta u_1 - c_2\delta v_1 - c_3\delta w_1 + c_1\delta u_2 + c_2\delta v_2 + c_3\delta w_2 \tag{6.4}$$

$$= (-c_1, -c_2, -c_3, c_1, c_2, c_3)\delta\mathbf{p} = \mathbf{r}^T\delta\mathbf{p}. \tag{6.5}$$

The vector $\mathbf{p}$ contains the global displacements, ordered in the standard way.

$$\mathbf{p}^T = (u_1, v_1, w_1, u_2, v_2, w_2) \tag{6.6}$$

Hence, the variation of the local strain producing displacement is

$$\delta p_\ell = \delta u_\ell = \mathbf{r}^T\delta\mathbf{p}. \tag{6.7}$$

6.2 Relations between local and global axial forces

Using the equivalence of virtual work in the local and global coordinate systems and using equation (6.7) it follows that

$$\delta\mathbf{p}_v^T\mathbf{q}_i = N_i\delta u_\ell = \delta p_{\ell v}^T q_{\ell i} = (\mathbf{r}^T\delta\mathbf{p}_v)^T q_{\ell i} = \delta\mathbf{p}_v^T\mathbf{r}q_{\ell i}, \tag{6.8}$$

where a subscript v denotes virtual quantities, a subscript ℓ denotes local quantities and $\mathbf{q}_i$ is the vector of global nodal forces for truss member i. The global nodal forces, $\mathbf{q}_i$, are ordered like global displacement vector $\mathbf{p}$. Now, since the $\delta\mathbf{p}_v$ are arbitrary it follows from (6.8) that

$$\mathbf{q}_i = q_{\ell i}\mathbf{r}, \tag{6.9}$$

where $q_{\ell i} = N_i = (AE/L_o)u_\ell$.

6.3 The variationally consistent tangent stiffness matrix

Taking the variation of (6.9) gives

$$\delta q_i = \delta q_{\ell i}\mathbf{r} + q_{\ell i}\delta\mathbf{r} = \mathbf{k}_{t1}\delta\mathbf{p} + \mathbf{k}_{t\sigma}\delta\mathbf{p}, \tag{6.10}$$

where $\mathbf{k}_{t1}$ is the standard transformed truss element tangent stiffness matrix at the global level and $\mathbf{k}_{t\sigma}$ is the initial stress or geometric stiffness matrix.

Differentiation of $q_{\ell i}$ leads to

$$\delta q_{\ell i} = \delta N_i = \delta\left(\frac{AE}{L_o}u_\ell\right) = \frac{AE}{L_o}\delta u_\ell = \frac{AE}{L_o}\mathbf{r}^T\delta\mathbf{p}. \tag{6.11}$$

Substituting (6.11) into the first part of (6.10) gives

$$\delta q_{\ell i}\mathbf{r} = \left(\frac{AE}{L_o}\mathbf{r}^T\delta\mathbf{p}\right)\mathbf{r} = \frac{AE}{L_o}\mathbf{r}\mathbf{r}^T\delta\mathbf{p}. \tag{6.12}$$

Comparing (6.12) with (6.10) it is evident that

$$\mathbf{k}_{t1} = \frac{AE}{L_o}\mathbf{r}\mathbf{r}^T, \tag{6.13}$$

where $\mathbf{r}\mathbf{r}^T$ is a tensor product somtimes written as $\mathbf{r}\otimes\mathbf{r}$. This is equivalent to saying that $\mathbf{r}\mathbf{r}^T$ is the square matrix with terms $a_{ij} = r_i r_j$.

The geometric stiffness comes about from the 2nd term in the first part of (6.10). Taking the variation of $\mathbf{r}$ yields

$$\delta\mathbf{r} = (-\delta c_1, -\delta c_2, -\delta c_3, \delta c_1, \delta c_2, \delta c_3)^T. \tag{6.14}$$

To determine $\delta c_1, \delta c_2, \delta c_3$ consider Figure 2. Observe that

$$\mathbf{B} = \mathbf{A} + \delta\mathbf{d}_{21} \quad \text{and} \quad \mathbf{A} = \mathrm{L}\mathbf{e}_1. \tag{6.15}$$

Unit vectors in the direction of $\mathbf{A}$ and $\mathbf{B}$ are

$$\mathbf{u}_A = \frac{A_1}{|\mathbf{A}|}\mathbf{i}_1 + \frac{A_2}{|\mathbf{A}|}\mathbf{i}_2 + \frac{A_3}{|\mathbf{A}|}\mathbf{i}_3 \quad \text{and} \quad \mathbf{u}_\mathrm{B} = \frac{\mathrm{B}_1}{|\mathbf{B}|}\mathbf{i}_1 + \frac{\mathrm{B}_2}{|\mathbf{B}|}\mathbf{i}_2 + \frac{\mathrm{B}_3}{|\mathbf{B}|}\mathbf{i}_3. \tag{6.16}$$

Recognize that the components of the unit vectors are the cosine direction angles for the respective vectors $\mathbf{A}$ and $\mathbf{B}$. Recognize also that for infinitesimal change $\delta\mathbf{d}_{21}$ from $\mathbf{A}$ to $\mathbf{B}$ the length relations are $|\mathbf{A}| \approx |\mathbf{B}| = L$. Hence, during a small change from vector $\mathbf{A}$ to vector $\mathbf{B}$ the first component of $\mathbf{u}_A$ changes to the first component of $\mathbf{u}_B$. This change *is* the variation in cosine direction angle c_1 (ie, δc_1). That is

$$\delta c_1 = \frac{B_1}{L} - \frac{A_1}{L} = \frac{A_1 + \delta u_2 - \delta u_1}{L} - \frac{A_1}{L} = \frac{\delta u_2 - \delta u_1}{L}. \tag{6.17}$$

The last equation can be expressed in the alternative form

$$\delta c_1 = \frac{1}{L}\,[-1\ 0\ 0\ 1\ 0\ 0]\begin{Bmatrix} \delta u_1 \\ \delta v_1 \\ \delta w_1 \\ \delta u_2 \\ \delta v_2 \\ \delta w_2 \end{Bmatrix} \tag{6.18}$$

Similarly,

$$\delta c_2 = \frac{\delta v_2 - \delta v_1}{L}, \tag{6.19}$$

$$\delta c_3 = \frac{\delta w_2 - \delta w_1}{L}. \tag{6.20}$$

The results (6.19) and (6.20) may be written in a form like (6.18) so that substitution into (6.14) gives

$$\delta \mathbf{r} = \frac{1}{L}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \delta u_1 \\ \delta v_1 \\ \delta w_1 \\ \delta u_2 \\ \delta v_2 \\ \delta w_2 \end{Bmatrix}. \tag{6.21}$$

Hence, the 2nd term in the first part of (6.10) becomes

$$q_{\ell i}\delta \mathbf{r} = \frac{q_{\ell i}}{L}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \delta u_1 \\ \delta v_1 \\ \delta w_1 \\ \delta u_2 \\ \delta v_2 \\ \delta w_2 \end{Bmatrix} = \mathbf{k}_{t\sigma}\delta \mathbf{p}, \tag{6.22}$$

where $\mathbf{k}_{t\sigma}$ is the geometric stiffness matrix. Finally, using (6.10), the variationally consistent tangent stiffness matrix is

$$\mathbf{k}_T = \mathbf{k}_{t1} + \mathbf{k}_{t\sigma}. \tag{6.23}$$

7 Co-rotational Algorithm - Load Control

The following is a load control algorithm for performing a co-rotational truss analysis. This is an implicit formulation which uses Newton-Raphson iterations at the global level to achieve equilibrium during each incremental load step. Material nonlinearities are not presently included in the algorithm. A program implementing this algorithm has been written in MATLAB and some representative results are provided. The algorithm proceeds as follows:

1. Define/initialize variables

- $\mathbf{F}$ = the total vector of externally applied global nodal forces
- $\mathbf{F}^{n+1}$ = the current externally applied global nodal force vector
- $\mathbf{N}$ = the vector of truss axial forces, axial force in truss element i is N_i
- $\mathbf{u}$ = the vector of global nodal displacements, initially $\mathbf{u} = \mathbf{0}$
- $\mathbf{X}$ = the vector of nodal x coordinates in the undeformed configuration
- $\mathbf{Y}$ = the vector of nodal y coordinates in the undeformed configuration
- $\mathbf{Z}$ = the vector of nodal z coordinates in the undeformed configuration
- $\mathbf{L}$ = the vector of truss element lengths based on current $\mathbf{u}$ using equation (3.2), L for truss i is L_i, save the initial lengths in a vector $\mathbf{L}_o$ by using equation (3.1).
- $\mathbf{c}$ = the vector of cosines for each truss element based on the current $\mathbf{u}$ using equations (5.1).
- $\mathbf{K} = \mathbf{K_M} + \mathbf{K_G}$, the assembled global tangent stiffness matrix
- $\mathbf{K}_s$ = the modified global tangent stiffness matrix to account for supports. Rows and columns associated with zero displacement dofs are set to zero and the diagonal position is set to 1.

2. **Start Loop** over load increments (for $n = 0$ to $ninc - 1$).
   (a) Calculate load factor $\lambda = 1/ninc$ and incremental force vector $d\mathbf{F} = \lambda \mathbf{F}$.
   (b) Calculate global stiffness matrix $\mathbf{K}$ based on current values of $\mathbf{c}, \mathbf{L}, \mathbf{L_o}$ and $\mathbf{N}$.
   (c) Modify $\mathbf{K}$ to account for supports and get $\mathbf{K}_s$.
   (d) Solve for the incremental global nodal displacements $d\mathbf{u} = \mathbf{K}_s^{-1} d\mathbf{F}$
   (e) Update global nodal displacements, $\mathbf{u}^{n+1} = \mathbf{u}^n + d\mathbf{u}$
   (f) Update the global nodal forces, $\mathbf{F}^{n+1} = \mathbf{F}^n + d\mathbf{F}$
   (g) Update $\mathbf{L}$ and $\mathbf{c}$
   (h) Calculate the vector of new internal truss element axial forces $\mathbf{N}^{n+1}$. For truss element i the axial force is $N_i^{n+1} = (A_i E / L_{oi})\, u_{\ell i}$.
   (i) Construct the vector of internal global forces $\mathbf{F}_{int}^{n+1}$ based on $\mathbf{N}^{n+1}$.
   (j) Calculate the residual $\mathbf{R} = \mathbf{F}_{int}^{n+1} - \mathbf{F}^{n+1}$ and modify the residual to account for the required supports.
   (k) Calculate the norm of the residual $R = \sqrt{\mathbf{R} \bullet \mathbf{R}}$
   (l) Iterate for equilibrium if necessary. Set up iteration variables.
      - Iteration variable = $k = 0$
      - $tolerance = 10^{-6}$
      - $maxiter = 100$
      - $\delta \mathbf{u} = \mathbf{0}$
      - $\mathbf{N}_{temp} = \mathbf{N}^{n+1}$

(m) **Start Iterations** while $R > tolerance$ and $k < maxiter$

i. $\mathbf{N}_{temp} = \mathbf{N}^{n+1}$

ii. Calculate the new global stiffness $\mathbf{K}$

iii. Modify the global stiffness to account for supports which gives $\mathbf{K}_s$

iv. Calculate the correction to $\mathbf{u}^{n+1}$, which is $\delta\mathbf{u}^{k+1} = \delta\mathbf{u}^{k} - \mathbf{K}_s^{-1}\mathbf{R}$, but note that $\mathbf{u}^{n+1}$ is not updated until all iterations are completed

v. Update $\mathbf{L}$ and $\mathbf{c}$ based on current $\mathbf{u}^{n+1} + \delta\mathbf{u}^{k+1}$

vi. Calculate the vector of new internal truss element axial forces $\mathbf{N}_{temp}^{k+1}$. For truss element i the axial force is $(N_{temp}^{k+1})_i = (A_i E/L_{oi})\, u_{\ell i}$.

vii. Construct the vector of internal global forces $\mathbf{F}_{int}^{n+1}$ based on $\mathbf{N}_{temp}^{k+1}$.

viii. Calculate the residual $\mathbf{R} = \mathbf{F}_{int}^{n+1} - \mathbf{F}^{n+1}$ and modify the residual to account for the required supports.

ix. $R = \sqrt{\mathbf{R} \bullet \mathbf{R}}$

x. Update iterations counter $k = k + 1$

(n) **End** of while loop iterations

3. Update variables to their final value for the current increment

   - $\mathbf{N}^{n+1} = \mathbf{N}_{temp}$
   - $\mathbf{u}_{final}^{n+1} = \mathbf{u}_{(0)}^{n+1} + \delta\mathbf{u}_{(k)}$

4. **End Loop** over load increments

8 Some Implementation Details–Calculating internal force vector

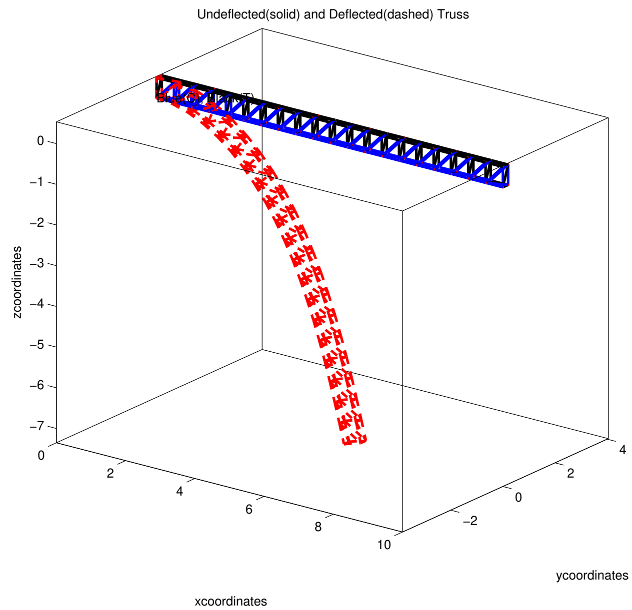
In the algorithms above the vector $\mathbf{N}$ is just a temporary vector to store the current axial force in each member of the truss (ie, $N_i = q_{\ell i}$ for truss member i). Hence, for a truss structure with n_m members the vector $\mathbf{N}$ is n_m by 1 in size. Using the ith row from $\mathbf{N}$, the internal force vector in global coordinates for truss member i is ((6.9))

$$\mathbf{q}_i = \mathbf{f}_{int}^{i} = N_i \mathbf{r} = q_{\ell i}\mathbf{r}, \tag{8.1}$$

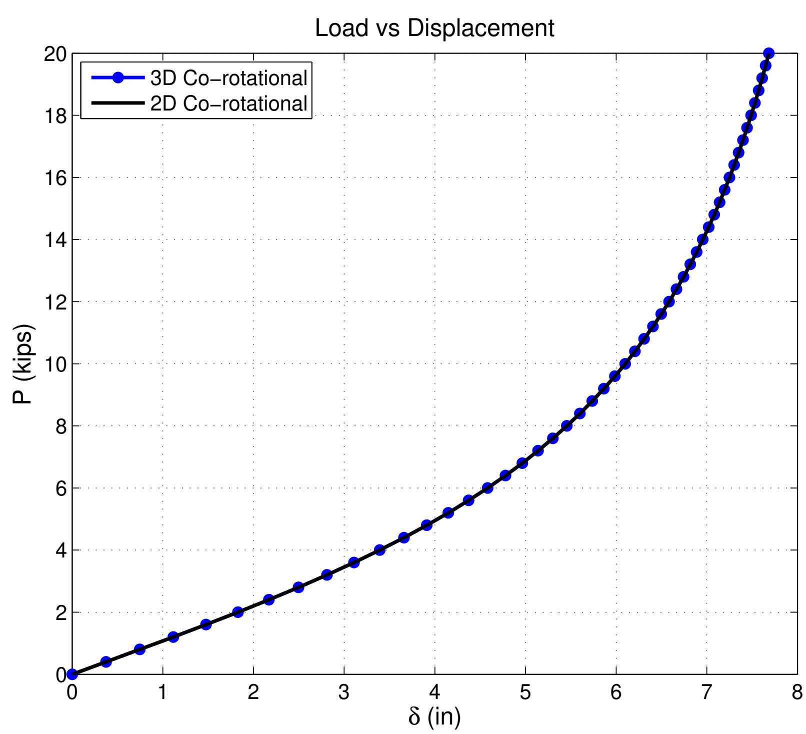
where it is understood that $\mathbf{r}$ is associated with truss member i like $\mathbf{q}_i$ and $q_{\ell i}$. Then by an appropriate assembly procedure, based on truss member degrees of freedom, the global internal force vector, $\mathbf{F}_{int}$ (size $3n_{nodes} \times 1$) is constructed. That is

$$\mathbf{F}_{int} = \mathop{\mathsf{A}}_{i=1}^{n_m} \mathbf{f}_{int}^{i}, \tag{8.2}$$

where A is the assembly operator [5] and n_m is the number of truss members in the structure.



(a)



(b)

Figure 3: Cantileverd Space Truss - Load Control: (a) Truss deflected shape, (b) Load versus free end vertical displacement (load and displacement are positive downwards).

9 Example – Cantilevered Space Truss With Lateral Restraint (Load Control)

The cantilevered space truss in this example is 10 inches long, 0.2 inches wide and 0.5 inches deep. The truss has two top chord members and two bottom chord members. For all side members (both nodes at y=0 or both nodes at y=0.2 inches) the area used is 0.05 in^2. For all other truss members the area is 0.0001 in^2. All members have a modulus of elasticity of E = 29000 ksi. Setting the material properties and areas as described creates a space truss with the same behavior as a 2D truss with all members of cross-sectional area of 0.1 in^2. The nodes at the support for the cantilever are restrained in the xyz directions. All nodes are restrained in the y direction, which forces the space truss to act like a 2D truss. This scenario is used to verify the current 3D formulation by comparing it to a working 2D formulation [9]. The 3D truss in its original and deflected configuration is shown in Figure 3a. The load displacement results for this 3D case are compared to the 2D case in Figure 3b. For the load displacement results a total load of 20 kips is applied in 50 equal load increments with an equilibrium tolerance at each load step of 10^{-6}. The 3D corotational results are identical to the 2D formulation.

10 Example – Three Bar Space Truss (Load Control)

If the reader tries to implement the 3D corotational truss formulation the following example problem given in [4] may be used as a check for correctness of results. The results are obtained using a program written in MATLAB using load control.

Consider the three bar space truss structure shown in Figure 4a. The bars are shown in the initial configuration. The plan view dimension is L = 500 cm. The initial height of the peak node is z_o = 20 cm. The modulus of elasticity E = 20500 kN/cm^2. The cross-sectional area of each bar is A = 6.53 cm^2. The applied load, P, is positive upwards and displacement, δ, is also positive upwards as shown.

If a total load of $P = -4.92$ kN (downwards) is applied (in 5 equal increments) to node 4 using load control the graph of Figure 4b is obtained, where the values of load and displacement are plotted. Results are shown for (i) the exact solution(given below) and (ii) a co-rotational truss solution using load control.

The exact solution for P as a function of δ is given by Crisfield [2] for a single truss bar. In this example, essentially 3 bars are acting together, hence the Crisfield solution is multiplied by three and is expressed as follows.

$$P(\delta) = \frac{3EA}{L^3}(z_o^2\delta + \frac{3}{2}z_o\delta^2 + \frac{1}{2}\delta^3). \tag{10.1}$$

The equation for $P(\delta)$ is used to plot the exact solution in Figure 4b (note that the above equation was derived in [2] with P and δ as positive upwards).

If the displacements continue this structure exibits snap through behavior. It is not possible for a load control scheme to trace the entire equilibrium path. This is seen if the maximum load is taken to $P = -8$ kN as shown in Figure 4c. In this case the load control scheme skips over the snap through path. In the following example the same truss is analyzed

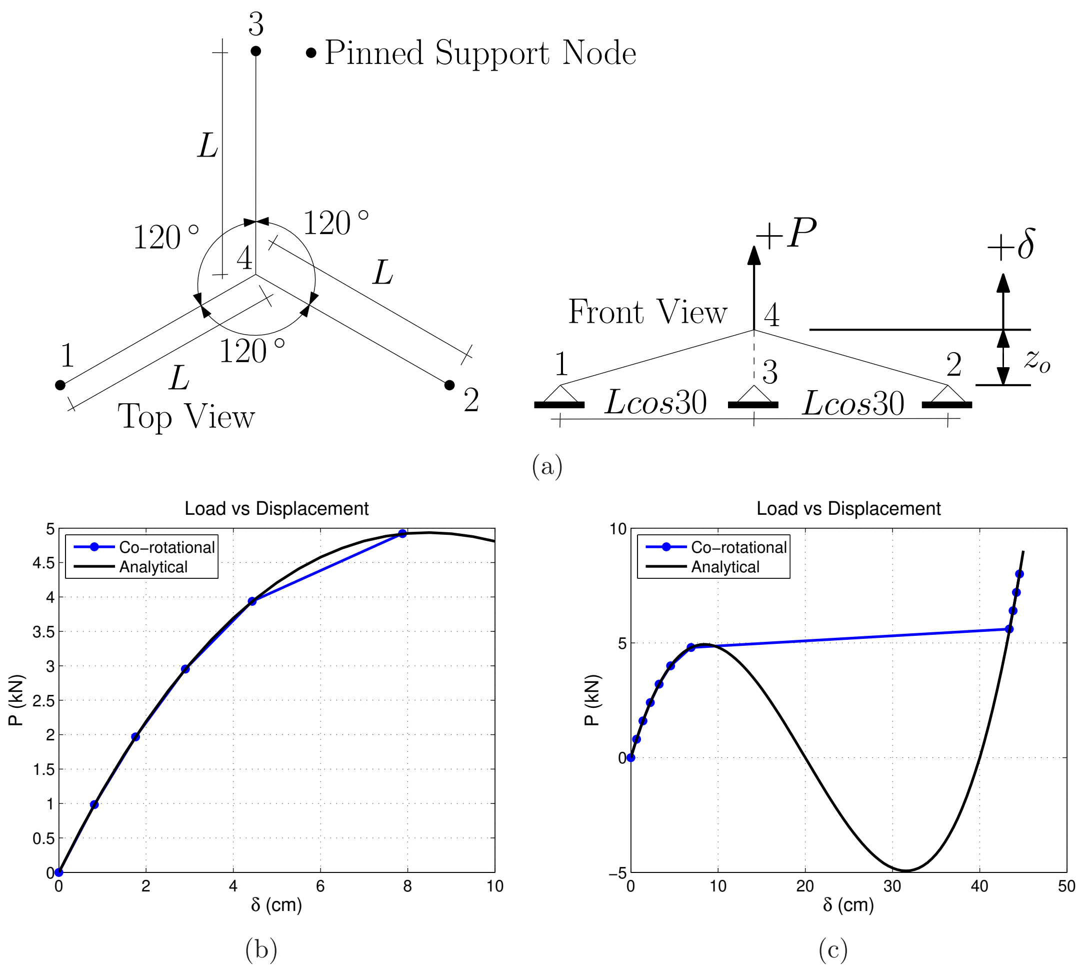


Figure 4: Three bar space truss: (a) initial configuration; (b) load P as a nonlinear function of displacement δ (load and displacement are plotted as positive downwards); (c) load P as a nonlinear function of displacement δ, load control does not trace the snap through equilibrium path.

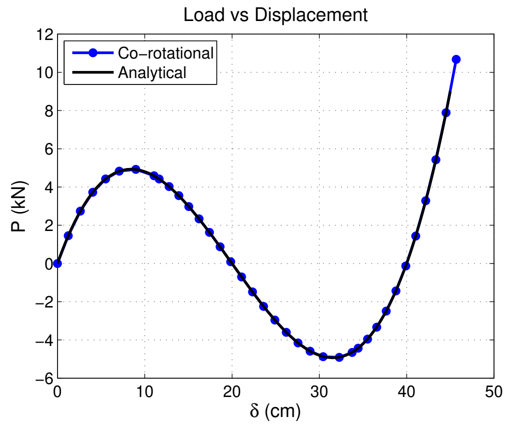


Figure 5: Three bar space truss: results for downward applied load P as a nonlinear function of displacement δ using Generalized Displacement Control (load and displacement are plotted as positive downwards).

using Generalized Displacement Control [8] and the equilibrium path is successfully traced even during snap through.

11 Example – Three bar space truss (Generalized Displacement Control)

Consider again the three bar space truss structure shown in Figure 4a. If a total load of $P = -10.67$ kN (downwards) is applied to node 4 using generalized displacement control the graph of Figure 5 is obtained. Results are shown for (i) the exact solution (see equation (10.1)) and (ii) a co-rotational truss solution using generalized displacement control [8]. In this case the entire equilibrium path is successfully followed. This demonstrates the importance of having a robust control scheme when solving geometrically nonlinear problems that may be subject to snap through and possibly snap back. Arc length control[1, 6, 2] is another popular method for following complicated equilibrium paths.

12 Example – Star Dome Space Truss

For problems of snap back and equilibrium path tracing (see Figure 7) methods such as generalized displacement control (see Yang et al [8]) or arc length control (see Crisfield [2] or Clarke and Hancock [1]) will be necessary. An excellent summary of many of the methods is given in the work by McGuire et al [6]. The following example uses generalized displacement control.

A common benchmark problem for space trusses is the star dome. This problem is discussed by Yang et al [8], Greco and Venturini [4], Wriggers [7], Crisfield [3] and many others. The geometry for the truss is shown in Figure 6. The results are obtained by using the following properties for all truss members: $A = 1cm^2$, $E = 1kN/cm^2$. The

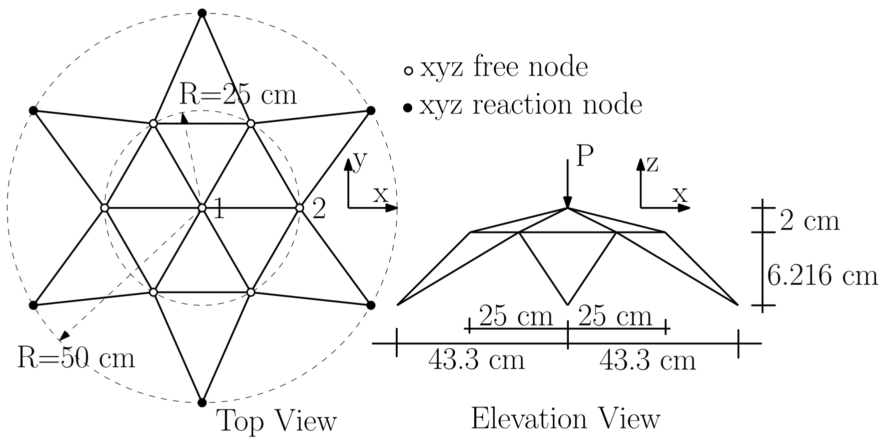


Figure 6: Model Geometry for Star Dome Space Truss

original and deflected shapes are shown in Figure 7a. The node 1, load versus vertical displacement, equilibrium path determined by the present corotational procedure is traced using generalized displacement control and is compared to the solution by Wriggers [7] in Figure 7b. It is evident from the figure that the corotational procedure is in excellent agreement with the solution provided by Wriggers. Figures 7cd show results for the same loading but for horizontal and vertical displacements at node 2 of the star dome. These results are in agreement with results which are given in [8].

13 Example – Lateral Torsional Buckling of Space Truss Beam

As a final example a cantilevered space truss beam is loaded with a vertical downward load at its free end until lateral torsional buckling takes place. The results are obtained by using the following properties for all truss members: $A = 1cm^2$, $E = 1kN/cm^2$. The final deflected shape of the cantilevered truss is shown in Figure 8a. The plot of load versus free end vertical displacement is shown in Figure 8b. The behavior is as expected. As the load is increase from zero the cantilever behaves in a linear fashion until lateral torsional buckling occurs and a loss of stiffness takes place. After further loading the cantilever is displaced downwards until it starts to stiffen again due to tension in the truss. This is as expected and illustrates the ability of the 3D corotational formulation to capture 3D buckling behavior.

14 Conclusion

A derivation and explanation of the ingredients of a 3D co-rotational truss formulation, in a small strain setting, is provided. The results of the formulation are compared to representative benchmark problems and are found to be in excellent agreement. A 3D corotational truss formulation is an effective technique for solving geometrically nonlinear problems.

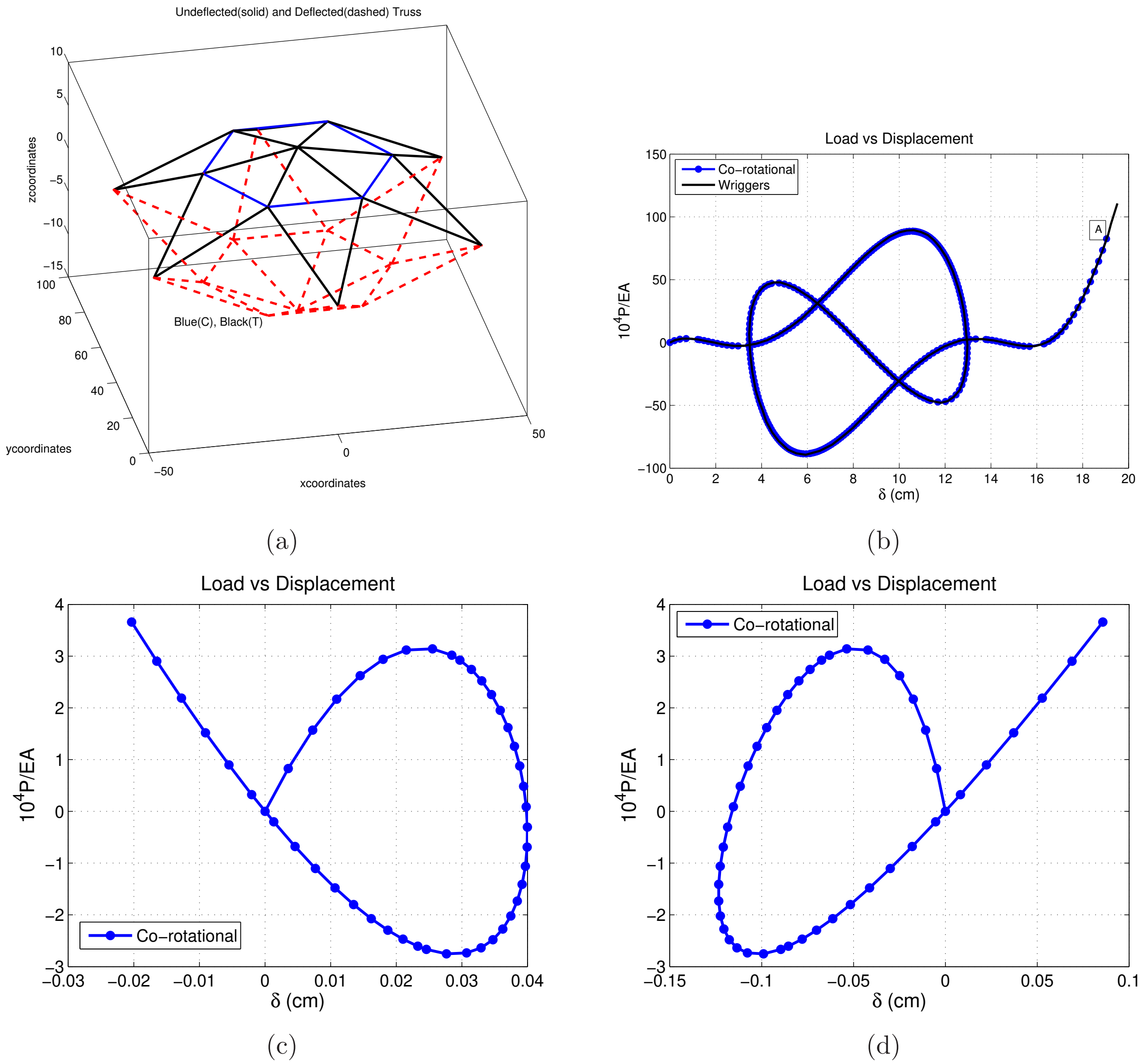
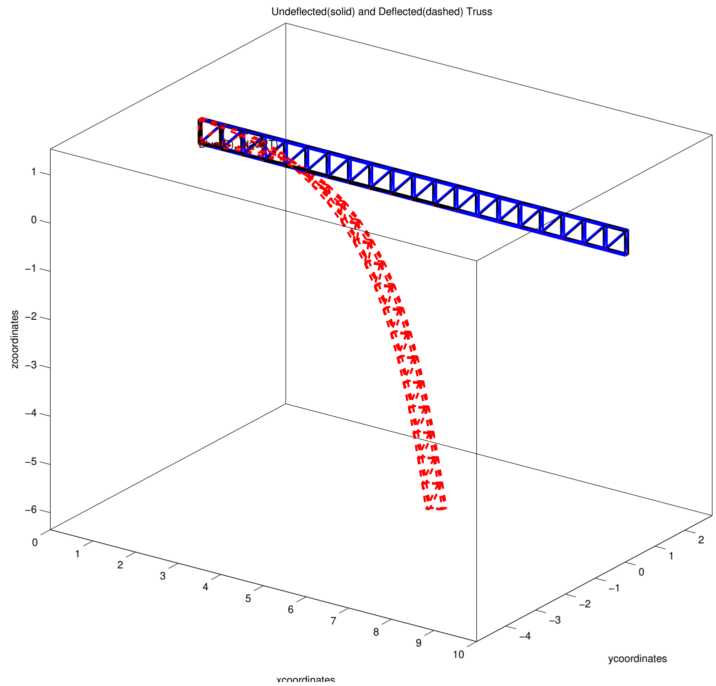
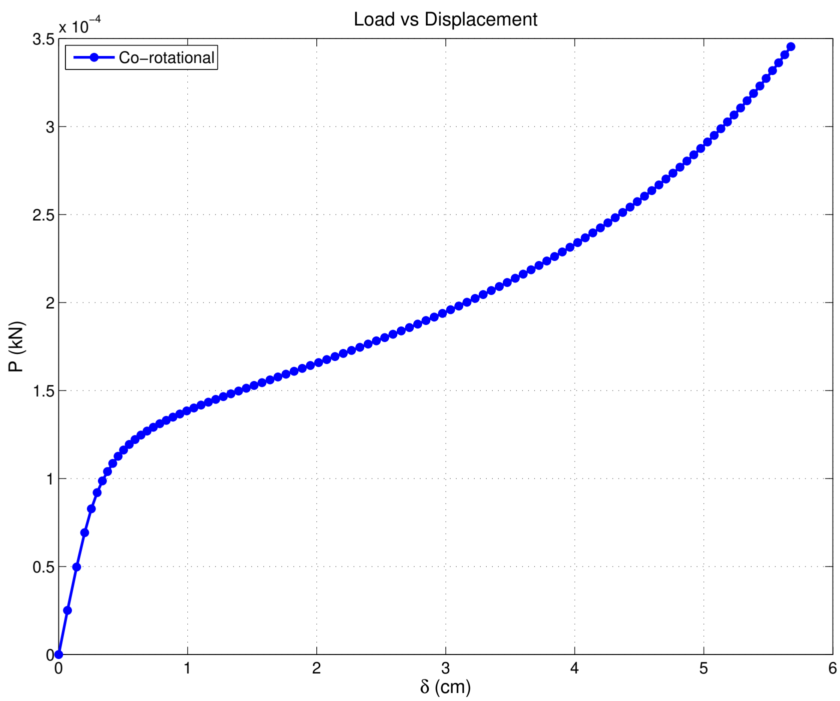


Figure 7: Star Dome Space Truss Analyzed By Generalized Displacement Control: (a) Truss deflected shape corresponding to point A on the load versus displacement plot, (b) Load versus node 1 vertical displacement (load and displacement are positive downwards), (c) Load versus node 2 horizontal displacement, (d) Load versus node 2 vertical displacement (load and displacement are positive downwards).



(a)



(b)

Figure 8: Lateral Torsional Buckling of a cantilevered space truss: (a) Truss deflected shape, (b) Load versus free end vertical displacement (load and displacement are positive downwards).

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