READ SECTION 1.4 PAGES 15-23 IN THE TEXTBOOK

READ CHAPTER 5 SECTIONS 1-5 AND PART OF CHAPTER 5 PAGES 180-182

3.10 PROPAGATION OF UNCERTAINTY (EQUIV. TO PAGES 180 → 182 IN TEXTBOOK.)

Often several quantities are measured, and the results of those measurements are used to calculate a desired result. For example, experimental values of density are usually determined by dividing the measured mass of a sample by the measured volume of that sample. Each measurement includes some uncertainty, and these uncertainties will create an uncertainty in the calculated result. What is that uncertainty?

Finding the uncertainty in a result due to uncertainties in the independent variables is called finding the *propagation of uncertainty*. For uncertainties in the independent variables, the procedure rests on a statistical theorem that is exact for a linear function y of several independent variables x_i with standard deviations σ_i; the theorem states that the standard deviation of y is

$$\sigma_y = \sqrt{\left(\frac{\partial y}{\partial x_1}\sigma_1\right)^2 + \left(\frac{\partial y}{\partial x_2}\sigma_2\right)^2 + \cdots + \left(\frac{\partial y}{\partial x_n}\sigma_n\right)^2} \tag{3.33}$$

Likewise, a calculated result y is a function of several independent measured variables, $\{x_1, x_2, \ldots, x_n\}$; for example, density is a function of mass and volume. Each measured value has some uncertainty, $\{u_1, u_2, \ldots, u_n\}$ and these uncertainties lead to an uncertainty in y, which we call u_y. To estimate u_y, we assume that each uncertainty is small enough that a first-order Taylor expansion of $y(x_1, x_2, \ldots, x_n)$ provides a reasonable approximation:

$$\begin{aligned} &y(x_1 + u_1, x_2 + u_2, \ldots, x_n + u_n) \\ &\qquad \approx y(x_1, x_2, \ldots, x_n) + \frac{\partial y}{\partial x_1}u_1 + \frac{\partial y}{\partial x_2}u_2 + \cdots + \frac{\partial y}{\partial x_n}u_n \end{aligned} \tag{3.34}$$

Under this approximation, y is a linear function of the independent variables. Now we can apply the theorem, assuming that uncertainties will behave much like standard deviations:

$$u_y = \sqrt{\left(\frac{\partial y}{\partial x_1}u_1\right)^2 + \left(\frac{\partial y}{\partial x_2}u_2\right)^2 + \cdots + \left(\frac{\partial y}{\partial x_n}u_n\right)^2} \quad (n:1) \tag{3.35}$$

THIS IS EQ 5.15 IN OUR TEXT

Here, all uncertainties must have the same odds and must be independent of each other. This approach was established by S. J. Kline and F. A. McClintock in 1953 [6].

The uncertainties, u_i, may be either systematic uncertainties (called bias uncertainty in some books) or random uncertainty (called precision uncertainty in some books). Normally, the systematic uncertainties and random uncertainties in y are propagated separately.

The overall uncertainty, U_y, is then calculated by combining B_y and P_y using Eq. ~~(3.4)~~. 5.2 IN OUR TEXT

FROM MECANICAL MEASUREMENTS BY BECKWITH

EXAMPLE 3.12

Suppose that y has the form

$$y = Ax_1 + Bx_2$$

and that the uncertainties in x_1 and x_2 are known with odds of $n : 1$. What is the uncertainty in y?

Solution

$$\frac{\partial y}{\partial x_1} = A$$

$$\frac{\partial y}{\partial x_2} = B$$

Using Eq. (3.35),

$$u_y = \sqrt{(Au_1)^2 + (Bu_2)^2} \qquad (n : 1) \tag{3.36}$$

For additive functions, the *absolute* uncertainties in each term are combined in root-mean-square (rms) sense.

EXAMPLE 3.13

Suppose that y has the form

$$y = A\,\frac{x_1^m x_2^n}{x_3^k}$$

and that the uncertainties in x_1, x_2, and x_3 are known with odds of $n : 1$. What is the uncertainty in y?

Solution

$$\frac{\partial y}{\partial x_1} = mA\,\frac{x_1^{(m-1)} x_2^n}{x_3^k}$$

$$\frac{\partial y}{\partial x_2} = nA\,\frac{x_1^m x_2^{(n-1)}}{x_3^k}$$

$$\frac{\partial y}{\partial x_3} = -kA\,\frac{x_1^m x_2^n}{x_3^{(k+1)}}$$

Using Eq. (3.35),

$$u_y = \sqrt{\left(mA\frac{x_1^{(m-1)}x_2^n}{x_3^k}u_1\right)^2 + \left(nA\frac{x_1^m x_2^{(n-1)}}{x_3^k}u_2\right)^2 + \left(-kA\frac{x_1^m x_2^n}{x_3^{(k+1)}}u_3\right)^2} \quad (n:1)$$

DIVIDE EACH TERM IN PARENTHESIS BY $y = A\dfrac{x_1^m x_2^n}{x_3^k}$ TO GET EQ. 3.37 & THE FRACTIONAL FORM

so that, for this case,

THIS IS A VERY USEFUL EXAMPLE ⟹

$$\frac{u_y}{y} = \sqrt{\left(m\frac{u_1}{x_1}\right)^2 + \left(n\frac{u_2}{x_2}\right)^2 + \left(k\frac{u_3}{x_3}\right)^2} \quad (n:1) \tag{3.37}$$

For multiplicative functions, the *fractional* uncertainties are combined in an rms sense. Note carefully the weighting factors, m, n, and k, in Eq. (3.37) and their sources.

Normally, each source of error is independent of the other sources. The errors will not all be of the same sign, nor will they all take on their maximum values simultaneously. For that reason, Eq. (3.35) combines the uncertainties in a root-mean-square sense. In some situations, however, various sources of uncertainty are not independent. Dependent errors should be added together, before combining them in the root-mean-square sense with other independent sources of error.

NOTE: USING FRACTIONAL UNCERTAINTY MAKES PROPAGATING UNCERTAINTY EASIER, i.e. LESS MESSY NUMERICALLY.