A Brief Introduction to Logic

Professor Dirac, a famous Applied Mathematician-Physicist, had a horse shoe over his desk. One day a student asked if he really believed that a horse shoe brought luck. Professor Dirac replied, “I understand that it brings you luck if you believe in it or not.”\(^1\)

A proposition is a statement that has a truth value, i.e. a statement that is either true or false. If \(p\) and \(q\) represent propositions, we can form the following additional propositions: \(p\) or \(q\), \(p\) and \(q\), not \(p\), if \(p\) then \(q\), \(p\) if and only if \(q\).

These compound propositions have the following truth values:

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The following statements mean the same thing:

\[
\begin{align*}
\text{if } p & \text{ then } q \\
\text{\(p\) implies } q & \\
\text{\(p\) only if } q & \\
\text{\(p\) is sufficient for } q & \\
\text{\(q\) is necessary for } p & \\
\text{\(q\) if } p & \\
\end{align*}
\]

The following statements mean the same thing:

\[
\begin{align*}
\text{\(p\) if and only if } q & \\
\text{\(p\) is necessary and sufficient for } q & \\
\end{align*}
\]

Statements that have identical truth values are said to be logically equivalent. For example, “\(p\) if and only if \(q\)” is logically equivalent to “\(\text{if } p \text{ then } q \text{ and if } q \text{ then } p\)”.

The statement “\(\text{if } p \text{ then } q\)” is called an implication. The converse of this implication is “\(\text{if } q \text{ then } p\)”, the inverse is “\(\text{if not } p \text{ then not } q\)”, and the contrapositive is “\(\text{if not } q \text{ then not } p\)”.

**Example 1** Find the contrapositive of the statement “\(\text{if } x = 2 \text{ then } x^2 = 4\)”.

**Theorem 1** An implication and its contrapositive are logically equivalent.

**Theorem 2** The following pairs of statements are logically equivalent:

\(^1\)http://www.naturalmath.com/jokes/joke9.html
The phrases “for each \(x\),” “for all \(x\),” and “for every \(x\)” are called universal quantifiers. Another type of quantifier is the existential quantifier, e.g., “there exists \(x\),” “for some \(x\),” and “there is an \(x\).”

Example 2 Discuss the following:

The Daily News published a story saying that one-half of the MP (Members of Parliament) were crooks. The Government took great exception to that and demanded a retraction and an apology. The newspaper responded the next day with an apology and reported that one-half of the MPs were not crooks.

Example 3 Find the negation of each statement.

1. All mathematics teachers are good people.
2. Some people are lazy.
3. For every real number \(x\), if \(x < 0\) then \(x^2 > 0\).
4. There exists a real number \(x\) satisfying \(x^2 + x < 0\).

The symbol “\(\forall\)” represents the universal quantifier. Thus, \(\forall x\) means “for all \(x\).” Similarly, the symbol “\(\exists\)” represents the existential quantifier, and \(\exists x\) means “there exists \(x\).”

Exercises

1. Give the precise negation of each of the following statements:
   (a) All crows are black.
   (b) All snakes are not poisonous.
   (c) Some problems are hard.
   (d) For all \(x\), if \(f(x) > 0\) then \(x + x^2 < 1\).

2. Give the contrapositive of each statement.
   (a) All crows are black. Hint: First write this in the “if...then” form.
   (b) If \(x^2 + 1 < 1\) then \(x\) is not a real number.
   (c) A sufficient condition for getting good grades is to be a genius.

3. Give counterexamples to show that the following are false.
   (a) All animals are carnivorous.
   (b) For all integers \(n\), \(n^2 + n + 41\) is prime.