A Brief Introduction to Logic

> *Professor Dirac, a famous Applied Mathematician-Physicist, had a horse shoe over his desk. One day a student asked if he really believed that a horse shoe brought luck. Professor Dirac replied,"I understand that it brings you luck if you believe in it or not."*[1]

A proposition is a statement that has a truth value, i.e. a statement that is either true or false. If p and q represent propositions, we can form the following additional propositions: p or q, p and q, not p, if p then q, p if and only if q.

These compound propositions have the following truth values:

p	q	p or q	p and q	not p	if p then q	p if and only if q
T	T	T	T	F	T	T
T	F	T	F	F	F	F
F	T	T	F	T	T	F
F	F	F	F	T	T	T

The following statements mean the same thing:

if p then q
p implies q
p only if q
p is sufficient for q
q is necessary for p
q if p

The following statements mean the same thing:

p if and only if q
p is necessary and sufficient for q

Statements that have identical truth values are said to be logically equivalent. For example, "p if and only if q" is logically equivalent to "if p then q and if q then p."

The statement "if p then q" is called an *implication*. The *converse* of this implication is "if q then p", the *inverse* is "if not p then not q," and the *contrapositive* is "if not q then not p."

Example 1 *Find the contrapositive of the statement "if $x = 2$ then $x^2 = 4$."*

Theorem 1 *An implication and its contrapositive are logically equivalent.*

Theorem 2 *The following pairs of statements are logically equivalent:*

[1] http://www.naturalmath.com/jokes/joke9.html

Statement 1	*Statement 2*
not(not p)	p
not (p or q)	*not p and not q*
not (p and q)	*not p or not q*

The phrases " for each x," " for all x," and " for every x" are called *universal quantifiers*. Another type of quantifier is the *existential quantifier*, e.g., "there exists x," "for some x", and "there is an x."

Example 2 *Discuss the following:*

> *The Daily News published a story saying that one-half of the MP (Members of Parliament) were crooks. The Government took great exception to that and demanded a retraction and an apology. The newspaper responded the next day with an apology and reported that one-half of the MPs were not crooks.*

Example 3 *Find the negation of each statement.*

1. *All mathematics teachers are good people.*
2. *Some people are lazy.*
3. *For every real number x, if $x < 0$ then $x^2 > 0$.*
4. *There exists a real number x satisfying $x^2 + x < 0$.*

The symbol "$\forall$" represents the universal quantifier. Thus, $\forall x$ means "for all x." Similarly, the symbol "$\exists$" represents the existential quantifier, and $\exists x$ means "there exists x."

Exercises

1. Give the precise negation of each of the following statements:
   (a) All crows are black.
   (b) All snakes are not poisonous.
   (c) Some problems are hard.
   (d) For all x, if $f(x) > 0$ then $x + x^2 < 1$.
2. Give the contrapositive of each statement.
   (a) All crows are black. *Hint: First write this in the "if...then" form.*
   (b) If $x^2 + 1 < 1$ then x is not a real number.
   (c) A sufficient condition for getting good grades is to be a genius.
3. Give counterexamples to show that the following are false.
   (a) All animals are carnivorous.
   (b) For all integers n, $n^2 + n + 41$ is prime.