The Black Point – Concluded

As the ship left the area of the Unspeakable Triangle, Dr. Zee's assistant turned to him and said, "The integration that you performed was most impressive, but I believe that we are traveling through a gravitational field, and if I remember my physics correctly, that force field is a conservative...."

"Zounds," shouted Dr. Zee, "a gravitational field is a conservative vector field. We'll calculate a potential function $f(x,y)$ for

$$\mathbf{F}(x,y) = \frac{-10^5}{(x^2+y^2)^{\frac{3}{2}}}\langle x,y\rangle = \left\langle \frac{-10^5 x}{(x^2+y^2)^{\frac{3}{2}}}, \frac{-10^5 y}{(x^2+y^2)^{\frac{3}{2}}} \right\rangle.$$

We want

$$\mathbf{F}(x,y) = \triangledown f(x,y) = \left\langle \frac{\partial}{\partial x} f(x,y), \frac{\partial}{\partial y} f(x,y) \right\rangle.$$

Thus

$$\frac{\partial}{\partial x} f(x,y) = \frac{-10^5 x}{(x^2+y^2)^{\frac{3}{2}}} \text{ and } \frac{\partial}{\partial y} f(x,y) = \frac{-10^5 y}{(x^2+y^2)^{\frac{3}{2}}}.$$

Integrating the first equation respect to x and holding y constant yields

$$f(x,y) = \int \frac{-10^5 x}{(x^2+y^2)^{\frac{3}{2}}}\, dx = \frac{10^5}{\sqrt{x^2+y^2}} + g(y).$$

This means that

$$\frac{\partial}{\partial y} f(x,y) = \frac{-10^5 y}{(x^2+y^2)^{\frac{3}{2}}} + g'(y).$$

Since this partial derivative is to equal the second component of $\mathbf{F}$, we see that $g'(y) = 0$ and $g(y) = k$, a constant. We might as well take $k = 0$ so that $f(x,y) = \frac{10^5}{\sqrt{x^2+y^2}}$," continued Dr. Zee. "The work done by the gravitational field in moving from (3,2) to $(3,-2)$ is $f(3,-2) - f(3,2) = 0$, and the done by the graviational field as the ship escapes starting at the point $(-1,0)$ is just the potential energy at the that point, namely $-f(-1,0) = -10^5$."

"Brilliant sir; you've done it again!" exclaimed his assistant.