

**Review and context:**  
**Sinusoidal Steady-State Analysis**

- We are moving from one special case (transient response with DC sources and step changes) to another special case (steady-state response with AC sinusoidal sources).
- Later, will learn the more sophisticated (but also more involved) method of Laplace transforms which can provide both transient *and* steady-state solutions for various stimuli. Phasor analysis can be seen as a special case where the stimulus is sinusoidal.
- Why sinusoidal sources?
  1. They're special! Sinusoids are *eigenfunctions* of linear systems (which includes any network of R's, L's, and C's).
  2. Fourier analysis says we can represent *any* signal as a combination of sinusoids, so one can leverage sinusoidal results to find circuit responses to non-sinusoidal signals.

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**Sinusoidal Steady-State Analysis**

- A transient response is still present because no source is eternal—it must be connected or “turned on” at some finite time in the past. But the transient response dies out quickly, so we don't lose much by ignoring it.

## Phasors

- A **phasor** is a complex number—a vector in the 2-D complex plane. It is used to represent an AC electrical quantity such as a voltage waveform or a current waveform:
  - Length of the phasor = peak value of the voltage or current
  - Angle of the phasor = phase angle of the voltage or current
- Phasors are used to represent the amplitude and phase relationships between two or more waveforms *with the same frequency*.
- Their primary purpose is to simplify the analysis of circuits involving sinusoidal excitation by providing an algebraic alternative to differential equations.

## Phasor examples

- Phasors don't need to encode the frequency of the waveforms, because this is understood to be the same as the frequency of the driving source.
- A typical phasor current is expressed as  $\mathbf{I} = I_M \angle \theta$ .  
For example,  $i(t) = 25 \cos(\omega t + 45^\circ)$  has the phasor representation  $\mathbf{I} = 25 \angle 45^\circ$ .
- A phasor voltage is written as  $\mathbf{V} = V_M \angle \theta$ .  
For example,  $v(t) = 15 \cos(\omega t + 120^\circ)$  has the phasor representation  $\mathbf{V} = 15 \angle 120^\circ$ .

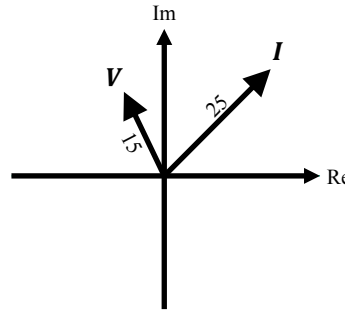
## Phasor examples

$$i(t) = 25 \cos(\omega t + 45^\circ) \quad \Leftrightarrow \quad \mathbf{I} = 25 \angle 45^\circ$$

$$v(t) = 15 \cos(\omega t + 120^\circ) \quad \Leftrightarrow \quad \mathbf{V} = 15 \angle 120^\circ$$

This diagram is a pictorial representation of the equations above:

The equations and the diagram convey the same information.



## Example Problems

### Useful trigonometric relationships:

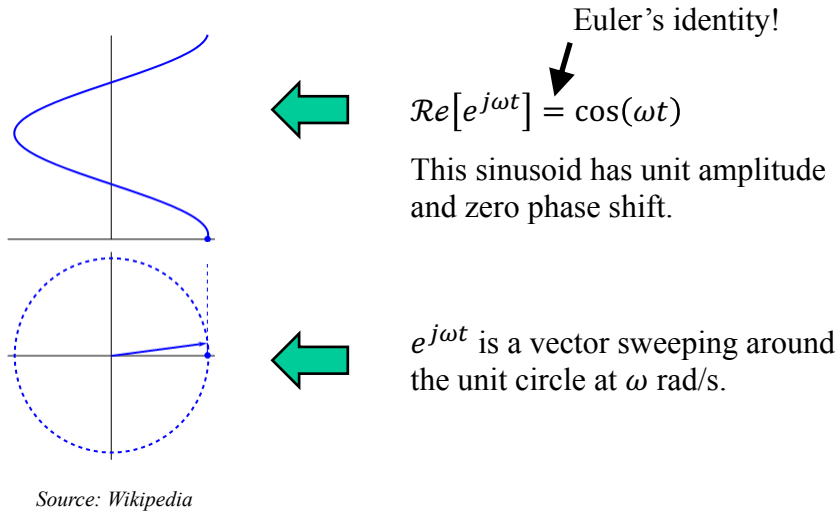
$$\begin{aligned} \sin(\omega t) &= \cos(\omega t - 90^\circ) & -\sin(\omega t) &= \cos(\omega t + 90^\circ) \\ \cos(\omega t) &= \sin(\omega t + 90^\circ) & -\cos(\omega t) &= \sin(\omega t - 90^\circ) \end{aligned}$$

### Express each of the following currents as a phasor:

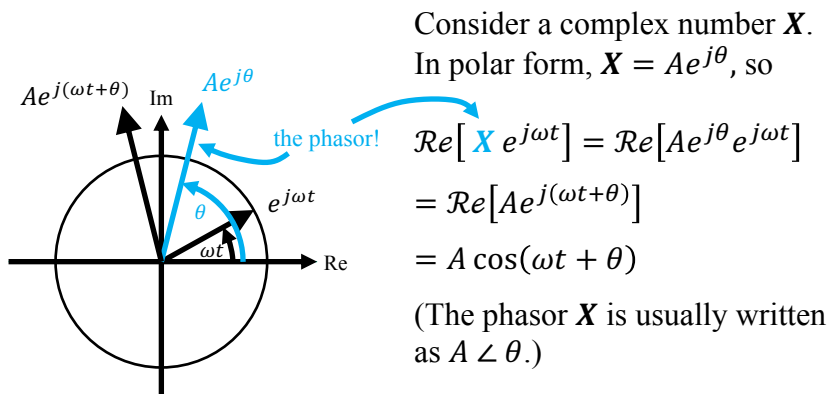
1.  $12\sin(400t + 110^\circ)\text{A}$
2.  $(-7\sin 800t - 3\cos 800t)\text{A}$
3.  $4\cos(200t - 30^\circ) - 5\cos(200t + 20^\circ)\text{A}$

1.  $12\sin(400t + 110^\circ)\text{A} = 12\cos(400t + 20^\circ)\text{A} = \mathbf{12\angle 20^\circ\text{A}}$
2.  $(-7\sin 800t - 3\cos 800t)\text{A} = 7\angle 90^\circ - 3\angle 0^\circ =$   
 $(0 + 7j) + (-3 + 0j) = -3 + 7j = \mathbf{7.616\angle 113.2^\circ\text{A}}$
3.  $4\cos(200t - 30^\circ) - 5\cos(200t + 20^\circ) = 4\angle -30^\circ - 5\angle 20^\circ =$   
 $(3.464 - 2j) - (4.70 + 1.71j) = -1.235 - 3.71j = \mathbf{3.91\angle -108^\circ\text{A}}$

## Formal description, building blocks



## Formal description of phasors



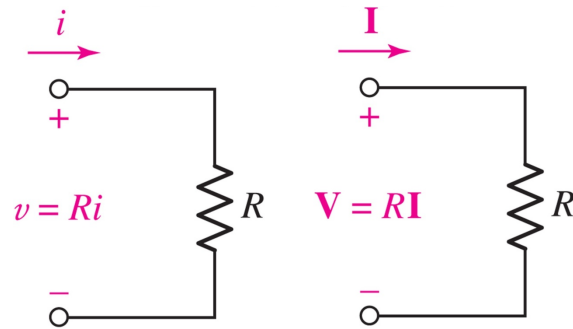
The “phasor domain” is a kind of shorthand where  $\mathbf{X}$  is used as a stand-in for the sinusoidal time function  $A \cos(\omega t + \theta)$ .  $\mathbf{X}$  is merely a complex number, but it represents the sinusoid having  $\mathbf{X}$ 's amplitude and phase. ( $\mathbf{X}$  does *not* define the sinusoid's frequency,  $\omega$ .)





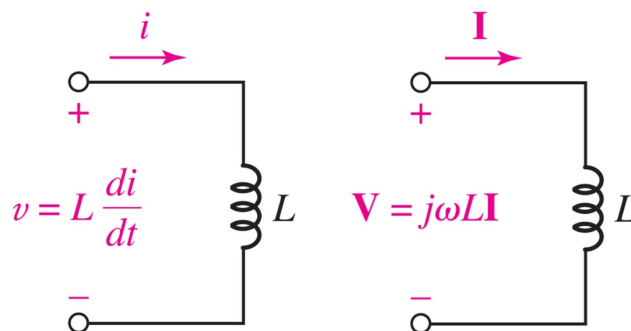
## Phasors: The Resistor

In the frequency domain, Ohm's Law takes the same form:



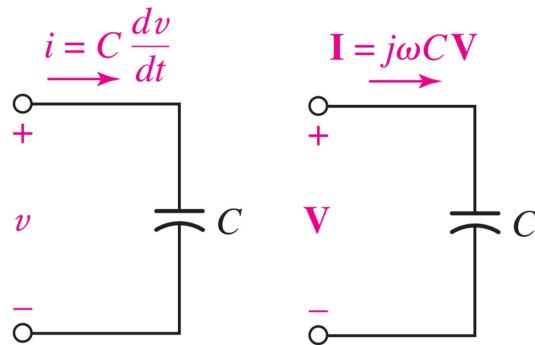
## Phasors: The Inductor

Differentiation in time becomes multiplication by  $j\omega$  in phasor form: (calculus becomes algebra).



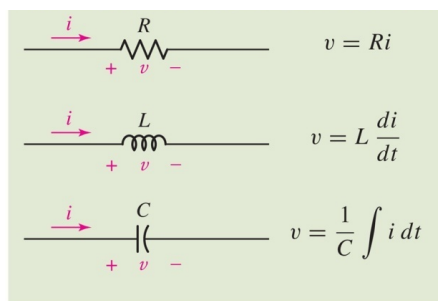
## Phasors: The Capacitor

Differentiation in time becomes multiplication by  $j\omega$  in phasor form: (calculus becomes algebra again).



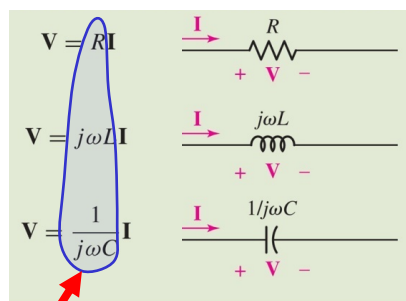
## Summary: Phasor Voltage/Current Relationships

Time Domain



Calculus (real numbers)

Frequency Domain



Algebra (complex numbers)

generalized resistance?



## Impedance

- The phasor relationship between voltage and current in a circuit is still defined by Ohm's law with resistance replaced by **impedance**, a frequency dependent form of resistance denoted as  $\mathbf{Z(j\omega)}$ .

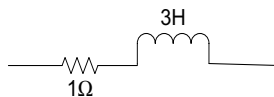
- We define impedance as  $\mathbf{Z} = \mathbf{V/I}$  or  $\mathbf{V} = \mathbf{IZ}$

$$\mathbf{Z}_R=R \quad \mathbf{Z}_L=j\omega L \quad \mathbf{Z}_C=1/j\omega C$$

- Impedance is a complex number (with units of ohms):
  - The real part of  $\mathbf{Z(j\omega)}$  is called **resistance**;
  - The imaginary part of  $\mathbf{Z(j\omega)}$  is called **reactance**.
  - Capacitors and inductors are purely reactive.  
Reactance is positive for inductors, negative for capacitors.
- Impedances in series or parallel can be combined using the same “resistor rules” that you learned in Chapter 3.

## Impedance Example

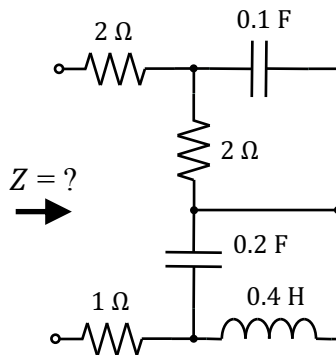
Find the equivalent impedance, in polar form, for the circuit below if  $\omega = 0.333$  rad/sec.



$$Z_{EQ} = R + j\omega L = 1 + j \cdot 3 \cdot \frac{1}{3} = 1 + j = \sqrt{2} \angle 45^\circ$$

## Example: Equivalent Impedance

Find the impedance of the network at  $\omega = 5$  rad/s.



Answer:

$$Z = (4 - j3)\ \Omega = 5 \angle -36.9^\circ\ \Omega$$

This impedance has both a *resistive* part ( $4\ \Omega$ ) and a *reactive* part ( $-j3\ \Omega$ ).

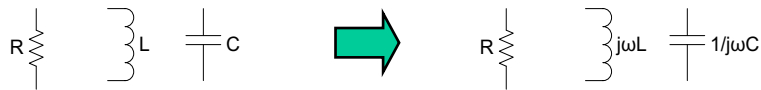
Because the reactive part is negative, EEs will sometimes say that the network “looks capacitive.” (Reactances  $> 0$  “look inductive.”)

## Circuit Analysis Using Phasors

- Techniques that can be used in circuit analysis with phasors:
  - Ohm’s law;
  - Kirchhoff’s voltage law (KVL);
  - Kirchhoff’s current law (KCL);
  - Source transformations;
  - Nodal analysis;
  - Mesh analysis;
  - Thévenin’s theorem;
  - Norton’s theorem;
  - Maximum power theorem.

## Circuit Analysis Procedure Using Phasors

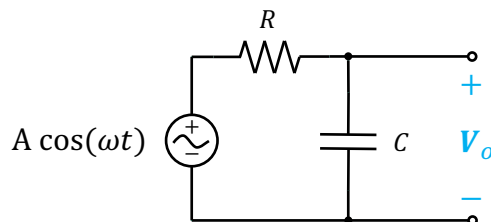
- Change the voltage/current sources into phasor form;
- Change R, L, and C values into phasor impedances;



- Use normal DC circuit analysis techniques but the values of voltage, current, and impedance can be complex numbers;
- Change back to the time-domain form if required.

## Example: Voltage divider using phasors

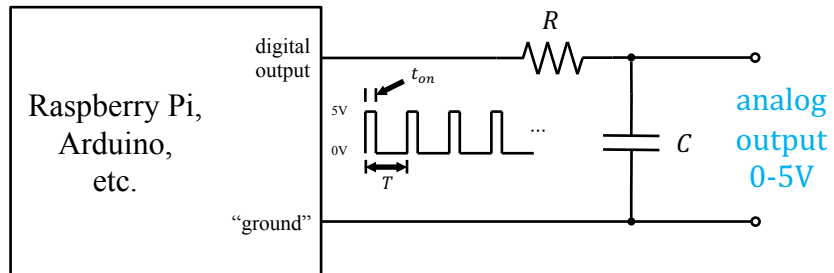
Find the magnitude and phase of  $V_o$ , and describe the circuit's behavior as a function of  $\omega$ .



$$\text{Answer: } V_o = \frac{A}{\sqrt{1+\omega^2 R^2 C^2}} \angle \tan^{-1}(-\omega RC)$$

This is a simple *low-pass filter*: At low frequencies (small  $\omega$ ) the output is almost the same as the input. At high frequencies (large  $\omega$ ) the output goes to zero.

## Practical application: “PWM DAC”



The digital signal can be decomposed into an average (DC) part of  $5 \left( \frac{t_{on}}{T} \right)$  V, plus sinusoids at the fundamental frequency  $1/T$  and higher frequencies.

If we choose  $RC \gg T$ , only the DC part gets through.