Section 6.2 Objective

- Be able to determine the natural and step responses of series RLC circuits.
Equations for Analysing the Natural Response of Parallel RLC Circuits

Characteristic equation

\[ s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \]

Neper, resonant, and damped frequencies

\[ \alpha = \frac{1}{2RC}, \quad \omega_0 = \sqrt{\frac{1}{LC}}, \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2} \]

Roots of the characteristic equation

\[ s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \]

\( \alpha^2 > \omega_0^2 \): overdamped

\[ v(t) = A_1e^{\omega_d t} + A_2e^{-\omega_d t}, \quad t \geq 0 \]

\[ v(0^+) = A_1 + A_2 = V_0 \]

\[ \frac{dv(0^+)}{dt} = \omega_1 A_1 - \omega_2 A_2 = \frac{1}{C} \left( -\frac{V_0}{R} - I_0 \right) \]

\( \alpha^2 < \omega_0^2 \): underdamped

\[ v(t) = B_1e^{-\alpha t} \cos(\omega_d t) + B_2e^{-\alpha t} \sin(\omega_d t), \quad t \geq 0 \]

\[ v(0^+) = B_1 = V_0 \]

\[ \frac{dv(0^+)}{dt} = -\alpha B_1 + \omega_1 B_2 = \frac{1}{C} \left( -\frac{V_0}{R} - I_0 \right) \]

\( \alpha^2 = \omega_0^2 \): critically damped

\[ v(t) = D_1e^{-\alpha t} + D_2e^{-\alpha t}, \quad t \geq 0 \]

\[ v(0^+) = D_1 = V_0 \]

\[ \frac{dv(0^+)}{dt} = -\alpha D_1 - \alpha D_2 = \frac{1}{C} \left( -\frac{V_0}{R} - I_0 \right) \]

(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)

Summary of Transient Responses

- \( R = 425 \) \( \cdot \), overdamped
- \( R = 200 \) \( \cdot \), critically damped
- \( R = 20 \) \( \cdot \), underdamped

Engr228 Zybooks Chapter 6.2 – Series RLC Circuits
Source - Free Series RLC Circuit

\[ i(t) + \frac{v_R(t)}{R} - \frac{v_C(t)}{C} - \frac{v_L(t)}{L} = 0 \]

\[ v_R + v_L + v_C = 0 \]

\[ i(t)R + L \frac{di(t)}{dt} + \frac{1}{C} \int i(t) \, dt = 0 \]

\[ L \frac{d^2i(t)}{dt^2} + R \frac{di(t)}{dt} + \frac{1}{C} i(t) = 0 \]

Comparing Series and Parallel RLC Circuits

### Parallel RLC

\[ C \frac{d^2v(t)}{dt^2} + \frac{1}{R} \frac{dv(t)}{dt} + \frac{1}{L} v(t) = 0 \]

\[ v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \]

\[ s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \]

\[ s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \]

\[ \alpha = \frac{1}{2RC} \quad \omega_0 = \frac{1}{\sqrt{LC}} \]

### Series RLC

\[ L \frac{d^2i(t)}{dt^2} + R \frac{di(t)}{dt} + \frac{1}{C} i(t) = 0 \]

\[ i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \]

\[ s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \]

\[ s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \]

\[ \alpha = \frac{R}{2L} \quad \omega_0 = \frac{1}{\sqrt{LC}} \]
Series RLC Circuit Solution

\[ \omega_0 = \frac{1}{\sqrt{LC}} \quad \alpha = \frac{R}{2L} \]

\[ s_i = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \]

\[ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \]

If:

\[ \alpha > \omega_0 \text{ (overdamped)}: \quad i(t) = A_1e^{\alpha t} + A_2e^{\alpha t} \]

\[ \alpha = \omega_0 \text{ (critically damped)}: \quad i(t) = e^{-\alpha t}(A_1 t + A_2) \]

\[ \alpha < \omega_0 \text{ (underdamped)}: \quad i(t) = e^{-\alpha t}(B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)) \]

Equations for Analysing the Natural Response of Series RLC Circuits

<table>
<thead>
<tr>
<th>Characteristic equation</th>
<th>[ s^2 + \frac{R}{L} s + \frac{1}{LC} = 0 ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Naper, resonant, and damped frequencies</td>
<td>[ \alpha = \frac{R}{2L}, \quad \omega_0 = \sqrt{\frac{1}{LC}}, \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2} ]</td>
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<tr>
<td>Roots of the characteristic equation</td>
<td>[ s_i = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} ]</td>
</tr>
</tbody>
</table>
| \[ \alpha^2 > \omega_0^2 \text{; overdamped} \] | \[ i(t) = A_1 e^{\alpha t} + A_2 e^{\alpha t}, \quad t \geq 0 \]
| \[ i(0^+) = A_1 + A_2 = I_0 \] | \[ \frac{di(0^+)}{dt} = s_1 A_1 + s_2 A_2 = \frac{1}{L} (-RI_0 - V_0) \] |
| \[ \alpha^2 < \omega_0^2 \text{; underdamped} \] | \[ i(t) = B_1 e^{-\omega_d t} \cos(\omega_d t) + B_2 e^{-\omega_d t} \sin(\omega_d t), \quad t \geq 0 \]
| \[ i(0^+) = B_1 = I_0 \] | \[ \frac{di(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 = \frac{1}{L} (-RI_0 - V_0) \] |
| \[ \alpha^2 = \omega_0^2 \text{; critically damped} \] | \[ i(t) = D_1 e^{-\alpha t} + D_2 e^{-\alpha t}, \quad t \geq 0 \]
| \[ i(0^+) = D_1 = I_0 \] | \[ \frac{di(0^+)}{dt} = -D_1 \alpha + \alpha D_2 = \frac{1}{L} (-RI_0 - V_0) \] |

(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)
Equations for Analysing the Step Response of Series RLC Circuits

<table>
<thead>
<tr>
<th>Characteristic equation</th>
<th>$s^2 + \frac{R}{L}s + \frac{1}{LC} = \frac{V}{L}$</th>
</tr>
</thead>
</table>
| Nper, resonant, and     | $\alpha = \frac{R}{2L}$  
| damped frequencies     | $\omega_0 = \sqrt{\frac{1}{LC}}$  
|                         | $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$  
| Roots of the           | $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$  
| characteristic         | $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$  
| equation                |                                                  |
| $\alpha^2 > \omega_0^2$| overdamped  
|                         | $v_c(t) = V_i + A_1 e^{\alpha t} + A_2 e^{\omega_0 t}, t \geq 0$  
|                         | $v_c(0^+) = V_i + A_1 + A_2 = V_0$  
|                         | $\frac{dv_c(0^+)}{dt} = s_1 A_1 + s_2 A_2 = \frac{I_0}{C}$  
| $\alpha^2 < \omega_0^2$| underdamped  
|                         | $v_c(t) = V_i + B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t, t \geq 0$  
|                         | $v_c(0^+) = V_i + B_1 = V_0$  
|                         | $\frac{dv_c(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 = \frac{I_0}{C}$  
| $\alpha^2 = \omega_0^2$ | critically damped  
|                         | $v_c(t) = V_i + D_1 e^{-\alpha t} + D_2 e^{-\alpha t}, t \geq 0$  
|                         | $v_c(0^+) = V_i + D_2 = V_0$  
|                         | $\frac{dv_c(0^+)}{dt} = -\alpha D_1 - \alpha D_2 = \frac{I_0}{C}$  

(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)

### Zybooks Response Summary

#### Series RLC

<table>
<thead>
<tr>
<th>Input: dc current with switch action $i(t) = 0$</th>
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<tr>
<td>Total Response</td>
</tr>
<tr>
<td>Overdamped ($\alpha &gt; \omega_0$)</td>
</tr>
<tr>
<td>$v_c(t) = A_1 e^{\alpha t} + A_2 e^{\omega_0 t} + \sigma v_c(0)$</td>
</tr>
</tbody>
</table>
| $A_1 = \frac{L}{\omega_0(L - \omega_0^2) + \sigma(L - \omega_0^2)}$  
| $s_1 = \omega_0$  
| $s_2 = -\omega_0$  
| $A_2 = \frac{L}{\omega_0(L - \omega_0^2) + \sigma(L - \omega_0^2)}$  
| $s_1 = -\omega_0$  
| Critically Damped ($\alpha = \omega_0$)          |
| $v_c(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t + \sigma v_c(0)$ |
| $B_1 = \frac{L}{\omega_0(L - \omega_0^2) + \sigma(L - \omega_0^2)}$  
| $B_2 = \frac{L}{\omega_0(L - \omega_0^2) + \sigma(L - \omega_0^2)}$  
| $s_1 = -\omega_0$  
| $s_2 = -\omega_0$  
| Underdamped ($\alpha < \omega_0$)                |
| $v_c(t) = c e^{-\alpha t} (B_1 \cos \omega_d t + B_2 \sin \omega_d t) + \sigma v_c(0)$ |
| $B_1 = \frac{L}{\omega_0(L - \omega_0^2) + \sigma(L - \omega_0^2)}$  
| $B_2 = \frac{L}{\omega_0(L - \omega_0^2) + \sigma(L - \omega_0^2)}$  
| $s_1 = -\omega_0$  
| $s_2 = -\omega_0$  

#### Parallel RLC

<table>
<thead>
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<th>Input: dc current with switch action $i(t) = 0$</th>
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<td>Total Response</td>
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<td>Overdamped ($\alpha &gt; \omega_0$)</td>
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<td>$i_L(t) = A_1 e^{\alpha t} + A_2 e^{\omega_0 t} + i_L(0)$</td>
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| $A_1 = \frac{L}{\omega_0(L - \omega_0^2) + \sigma(L - \omega_0^2)}$  
| $s_1 = \omega_0$  
| $s_2 = -\omega_0$  
| $A_2 = \frac{L}{\omega_0(L - \omega_0^2) + \sigma(L - \omega_0^2)}$  
| $s_1 = -\omega_0$  
| Critically Damped ($\alpha = \omega_0$)          |
| $i_L(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t + i_L(0)$ |
| $B_1 = \frac{L}{\omega_0(L - \omega_0^2) + \sigma(L - \omega_0^2)}$  
| $B_2 = \frac{L}{\omega_0(L - \omega_0^2) + \sigma(L - \omega_0^2)}$  
| $s_1 = -\omega_0$  
| $s_2 = -\omega_0$  
| Underdamped ($\alpha < \omega_0$)                |
| $i_L(t) = c e^{-\alpha t} (B_1 \cos \omega_d t + B_2 \sin \omega_d t) + i_L(0)$ |
| $B_1 = \frac{L}{\omega_0(L - \omega_0^2) + \sigma(L - \omega_0^2)}$  
| $B_2 = \frac{L}{\omega_0(L - \omega_0^2) + \sigma(L - \omega_0^2)}$  
| $s_1 = -\omega_0$  
| $s_2 = -\omega_0$  

#### Auxiliary Relations

- $\alpha = \frac{R}{2L}$  
- $\omega_0 = \sqrt{\frac{1}{LC}}$  
- $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$  
- $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$  
- $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$  
- $\frac{\omega_0}{\sqrt{LC}}$  
- $\omega_0 = \sqrt{\omega_0^2 - \alpha^2}$  
- $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$  
- $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$  

Engr228 Zybooks Chapter 6.2 – Series RLC Circuits
Textbook Problem 8.50 (Nilsson 11th)

The circuit contains no initial energy. Find $v_o(t)$ for $t \geq 0$.

$$v_o(t) = 16 - 16e^{-400t}\cos 300t - 21.33e^{-400t}\sin 300t \ V$$

Zybooks Participation Exercise 6.5.3
**Summary: Solving RLC Circuits**

1. Identify the series or parallel RLC circuit;
2. Find $\alpha$ and $\omega_0$;
3. Determine whether the circuit is overdamped, critically damped, or underdamped;
4. Assume a solution (natural response + forced response):

   \[ A_1 e^{\alpha t} + A_2 e^{\alpha t} + V_f \quad \text{Overdamped} \]

   \[ A_1 t e^{\alpha t} + A_2 e^{\alpha t} + V_f \quad \text{Critically damped} \]

   \[ e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t) + V_f \quad \text{Underdamped} \]

5. Find $A$, $B$, and $V_f$ using initial and final conditions.

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**Equations for Analysing the Natural Response of Parallel RLC Circuits**

\[ x^2 + \frac{1}{RC} x + \frac{1}{LC} = 0 \]

**Characteristic equation**

\[ \alpha = \frac{1}{2RC} \quad \omega_0 = \frac{1}{\sqrt{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \omega^2} \]

**Nepers, resonant, and damped frequencies**

\[ s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \]

**Roots of the characteristic equation**

\[ v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}, \quad t \geq 0 \]

\[ v(0^+) = A_1 + A_2 = V_0 \]

\[ \frac{dv(0^+)}{dt} = s_1 A_1 + s_2 A_2 = \frac{1}{C} \left( -\frac{V_0}{R} - I_0 \right) \]

\[ \omega^2 > \omega_0^2 : \text{overdamped} \]

\[ v(t) = B_1 e^{\omega_d t} \cos \omega_d t + B_2 e^{\omega_d t} \sin \omega_d t, \quad t \geq 0 \]

\[ v(0^+) = B_1 = V_0 \]

\[ \frac{dv(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 = \frac{1}{C} \left( -\frac{V_0}{R} \right) \]

\[ \frac{dv(0^+)}{dt} = B_1 - \alpha B_2 = \frac{1}{C} \left( -\frac{V_0}{R} - I_0 \right) \]

\[ \omega^2 < \omega_0^2 : \text{underdamped} \]

\[ \omega^2 = \omega_0^2 : \text{critically damped} \]

(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)
### Equations for Analysing the Step Response of Parallel RLC Circuits

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<td>( \alpha^2 &gt; \omega_0^2 ): overdamped</td>
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(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)

### Section 6.2 Summary

- Showed how to determine the natural and step responses of series RLC circuits.