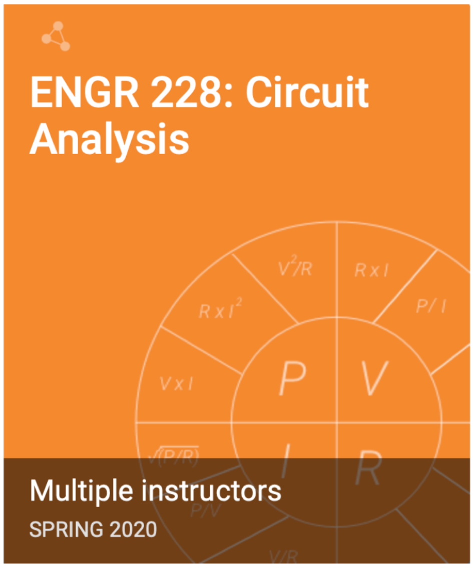


Chapter 6

RLC Circuits

Engr228 - Circuit Analysis

Spring 2020

Dr Curtis Nelson

Chapter 6 Objectives

- Be able to determine the natural and the step response of parallel RLC circuits;
- Be able to determine the natural and the step response of series RLC circuits.

First-Order RL and RC Circuit Review

- Transient, natural, or homogeneous response:
  - Fades over time;
  - Resists change.
- Forced, steady-state, particular response:
  - Follows the input;
  - Independent of time passed.
- The total response will be:

$$i_L(t) = i_L(\infty) + (i_L(0) - i_L(\infty))e^{-\left(\frac{R}{L}\right)t}$$

$$v_c(t) = v_c(\infty) + (v_c(0) - v_c(\infty))e^{-\frac{t}{RC}}$$

RLC Circuits

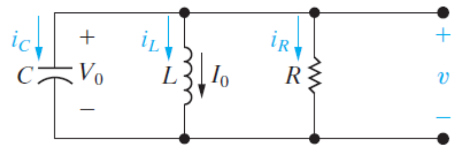
- RLC circuits contain **both** an inductor and a capacitor;
- These circuits have a wide range of applications including oscillators, frequency filters, flight simulation, modeling automobile suspensions, and more;
- The response of RLC circuits with DC sources and switches will consist of a natural response and a forced response:

$$v(t) = v_f(t)+v_n(t)$$

The complete response must satisfy both the **initial conditions** and the **final conditions** of the forced response.

Source-Free Parallel RLC Circuits

We will first study the natural response of second-order circuit by looking at a source-free parallel RLC circuit:



$$i_R + i_L + i_C = 0$$

$$\frac{v(t)}{R} + \frac{1}{L}\int v(t)dt + C\frac{dv(t)}{dt} = 0$$

$$C\frac{d^2v(t)}{dt^2} + \frac{1}{R}\frac{dv(t)}{dt} + \frac{1}{L}v(t) = 0$$

⇨ Second-order Differential equation

Source-Free Parallel RLC Circuits

This second-order differential equation can be solved by assuming the form of a solution:

$$v(t) = Ae^{st}$$

$$C\frac{d^2v(t)}{dt^2} + \frac{1}{R}\frac{dv(t)}{dt} + \frac{1}{L}v(t) = 0$$

$$CAs^2e^{st} + \frac{1}{R}Ase^{st} + \frac{1}{L}Ae^{st} = 0$$

$$Ae^{st}(Cs^2 + \frac{1}{R}s + \frac{1}{L}) = 0$$

which means $Cs^2 + \frac{1}{R}s + \frac{1}{L} = 0$

- This is known as the ***characteristic equation***.

Source-Free Parallel RLC Circuits

Using the quadratic formula, we get

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

Define resonant frequency: $\omega_0 = \frac{1}{\sqrt{LC}}$

Define damping factor: $\alpha = \frac{1}{2RC}$
(neper frequency)

Then: $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

Second-Order Differential Equation Solution

We will now divide the circuit response into three cases according to the sign of the term under the radical.

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

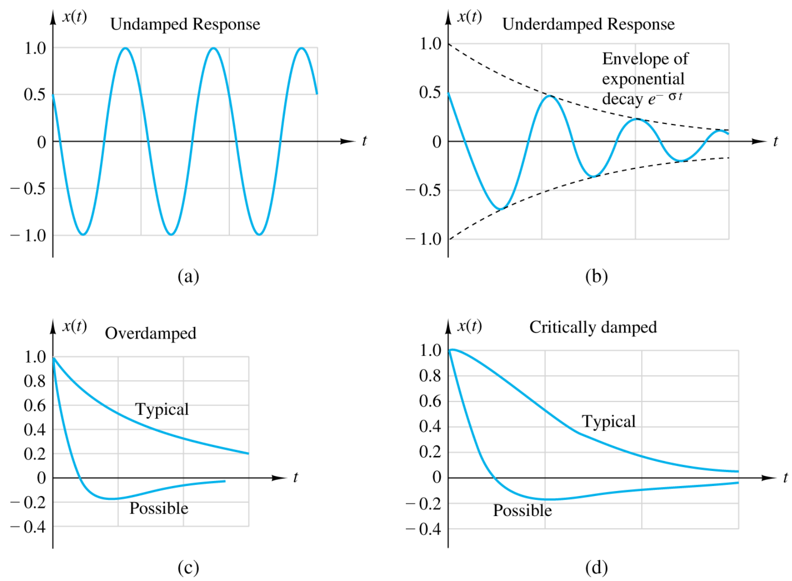
$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$\alpha > \omega_0$ (overdamped): $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

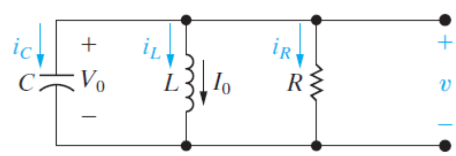
$\alpha = \omega_0$ (critically damped): $v(t) = A_1 t e^{st} + A_2 e^{st}$

$\alpha < \omega_0$ (underdamped): $v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$

Types of Circuit Responses



Solving an Overdamped RLC Circuit

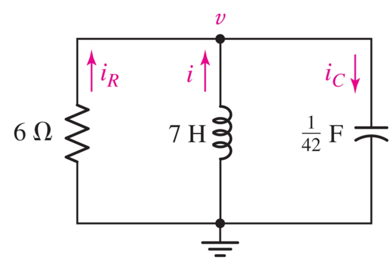


$$v_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- We need two equations to solve the second order circuit:
  - Evaluate v_C at t = 0^+ $\quad v_C(0^+) = A_1 + A_2$
  - Evaluate $\frac{dv}{dt}$ at t = 0^+ $\quad A_1 s_1 + A_2 s_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$
- Note the second equation is equivalent to writing a node equation and evaluating at t = 0^+.

Overdamped Example

Find v(t) in the circuit at the right. Ignore the current arrows.



Given initial conditions:
$v_c(0) = 0$, $i_L(0) = -10A$

$$\alpha = \frac{1}{2RC} = 3.5 \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{6}$$

$\alpha > \omega_0$ therefore this is an overdamped case

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \qquad s_1 = -1,\ s_2 = -6$$

Overdamped Case - continued

The solution is of the form: $v(t) = A_1 e^{-t} + A_2 e^{-6t}$

Use initial conditions to find A_1 and A_2

From $v_c(0) = 0$ at t = 0:

$$v(0) = 0 = A_1 e^0 + A_2 e^0 = A_1 + A_2$$

From KCL taken at t = 0:

$$i_R + i_L + i_C = 0$$

$$\frac{v(0)}{R} + (-10) + C\left.\frac{dv(t)}{dt}\right|_{t=0} = 0$$

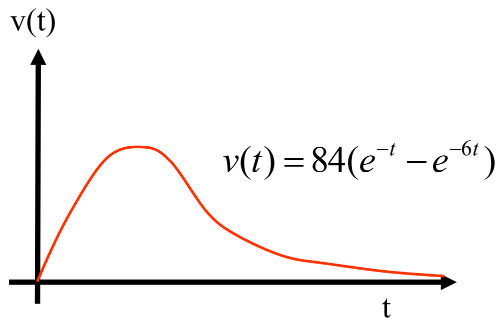
$$\frac{0}{R} + (-10) + \frac{1}{42}\left(-A_1 e^{-t} - 6A_2 e^{-6t}\right)\Big|_{t=0} = 0$$

$$\left(-A_1 - 6A_2\right) = 420$$

Overdamped Case - continued

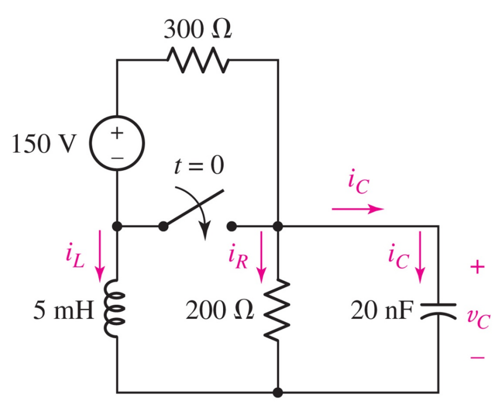
Solving the two equations we get $A_1 = 84$ and $A_2 = -84$
The solution is:

$$v(t) = 84e^{-t} - 84e^{-6t} = 84(e^{-t} - e^{-6t})V$$



Example: Overdamped RLC Circuit

Find $v_C(t)$ for t > 0.



$v_C(t) = 80e^{-50{,}000t} - 20e^{-200{,}000t}$ *V for t > 0*

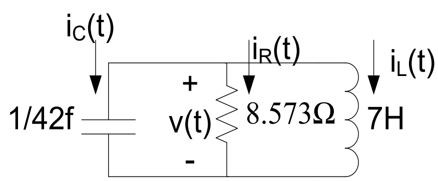
Equations for Analysing the Natural Response of Parallel *RLC* Circuits

Characteristic equation	$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$
Neper, resonant, and damped frequencies	$\alpha = \frac{1}{2RC} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Roots of the characteristic equation	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
$\alpha^2 > \omega_0^2$: overdamped	$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}, t \geq 0$ $v(0^+) = A_1 + A_2 = V_0$ $\frac{dv(0^+)}{dt} = s_1 A_1 + s_2 A_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$
$\alpha^2 < \omega_0^2$: underdamped	$v(t) = B_1 e^{-\alpha t}\cos \omega_d t + B_2 e^{-\alpha t}\sin \omega_d t, t \geq 0$ $v(0^+) = B_1 = V_0$ $\frac{dv(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$
$\alpha^2 = \omega_0^2$: critically damped	$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}, t \geq 0$ $v(0^+) = D_2 = V_0$ $\frac{dv(0^+)}{dt} = D_1 - \alpha D_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$

(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)

Critically Damped Case ($\alpha = \omega_0$)

Find v(t) in the circuit at the right.



Given initial conditions:
$v_c(0) = 0$, $i_L(0) = -10A$

$$\alpha = \frac{1}{2RC} = \omega_0 = \frac{1}{\sqrt{LC}} = 2.45$$

Critically damped when $\alpha = \omega_0$ $\qquad$ $s_1 = s_2 = -2.45$

The complete solution is of the form:

$$v(t) = A_1 t e^{st} + A_2 e^{st}$$

Critically Damped Case - continued

Use initial conditions to find A_1 and A_2

From $v_c(0) = 0$ at $t = 0$:

$$v(0) = 0 = A_1(0)e^0 + A_2e^0 = A_2$$

Therefore $A_2 = 0$ and the solution is reduced to $v(t) = A_1 te^{-2.45t}$

Find A_1 from KCL at $t = 0$:

$$i_R + i_L + i_C = 0$$

$$\frac{v(0)}{R} + (-10) + C\left.\frac{dv(t)}{dt}\right|_{t=0} = 0$$

$$\frac{0}{R} + (-10) + \frac{1}{42}\left(A_1 t(-2.45)e^{-2.45t} + A_1 e^{-2.45t}\right)\Big|_{t=0} = 0$$

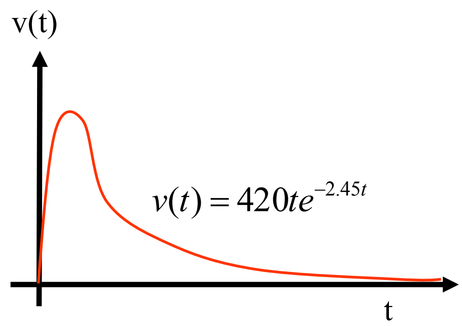
$$-10 + \frac{1}{42}(A_1) = 0$$

Critically Damped Case - continued

Solving the equation: $A_1 = 420$

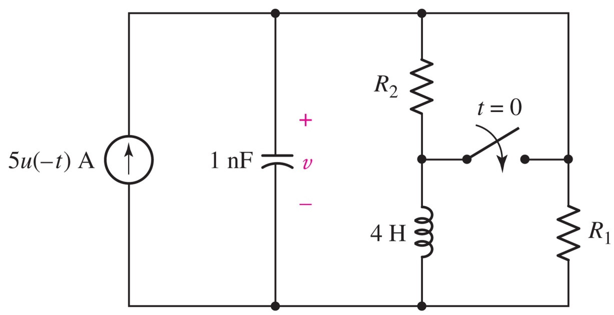
The solution is:

$$v(t) = 420te^{-2.45t}V$$



Critically Damped Example

Find R_1 such that the circuit is critically damped for $t > 0$ and R_2 such that $v(0)=2V$.



Answer: $R_1 = 31.63\ k\Omega$, $R_2=0.4\Omega$

Underdamped Case ($\alpha < \omega_0$)

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

For the underdamped case, the term inside the bracket will be negative and s will be a complex number.

Define $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

Then $s_{1,2} = -\alpha \pm j\omega_d$

$$v(t) = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t}$$
$$v(t) = e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t})$$

Underdamped Case - continued

$$v(t) = e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t})$$

Using Euler's Identity $e^{j\theta} = \cos\theta + j\sin\theta$

$$v(t) = e^{-\alpha t}(A_1 \cos\omega_d t + jA_1 \sin\omega_d t + A_2 \cos\omega_d t - jA_2 \sin\omega_d t)$$

$$v(t) = e^{-\alpha t}((A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t)$$

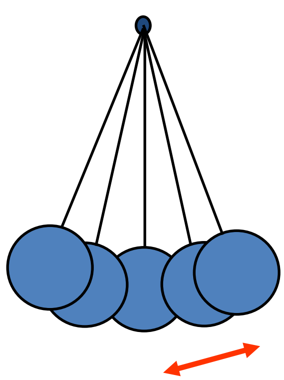
$$v(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$$

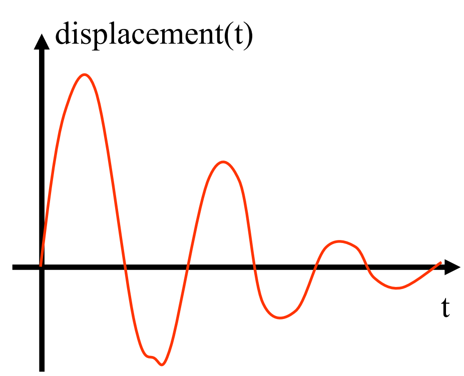
Looking at the magnitude:

$$v(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$$

Mechanical Analogue

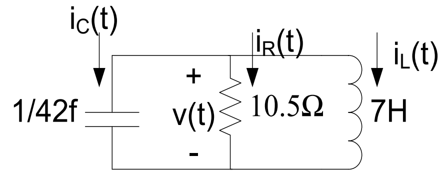
A pendulum is an example of an underdamped second-order mechanical system.





Underdamped Case - Example

Find v(t) in the circuit at the right.



Given initial conditions:
$v_c(0) = 0$, $i_L(0) = -10A$

$$\alpha = \frac{1}{2RC} = 2 \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{6}$$

$\alpha < \omega_0$ therefore, this is an underdamped case

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{2}$$

v(t) is of the form: $v(t) = e^{-2t}(B_1 \cos\sqrt{2}t + B_2 \sin\sqrt{2}t)$

Underdamped Case - continued

Use initial conditions to find B_1 and B_2

From $v_c(0) = 0$ at t = 0:

$$v(0) = e^0(B_1 \cos 0 + B_2 \sin 0) = B_1$$

Therefore $B_1 = 0$ and the solution is reduced to

$$v(t) = e^{-2t}(B_2 \sin\sqrt{2}t)$$

Find B_2 from KCL at t = 0:

$$i_R + i_L + i_C = 0$$

$$\frac{v(0)}{R} + (-10) + C\left.\frac{dv(t)}{dt}\right|_{t=0} = 0$$

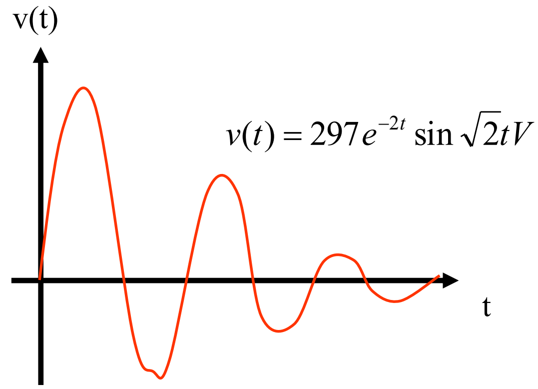
$$\frac{0}{R} + (-10) + \frac{1}{42}\left.\left(\sqrt{2}B_2 e^{-2t}\cos\sqrt{2}t - 2B_2 e^{-2t}\sin\sqrt{2}t\right)\right|_{t=0} = 0$$

$$-10 + \frac{1}{42}(\sqrt{2}B_2) = 0$$

Underdamped Case - continued

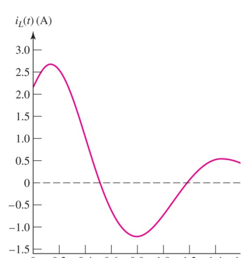
Solving: $B_2 = 210\sqrt{2} = 297$

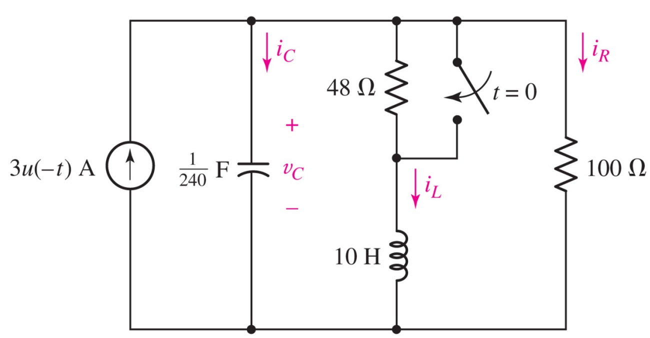
The solution is: $v(t) = 297e^{-2t}\sin\sqrt{2}tV$



Underdamped Example

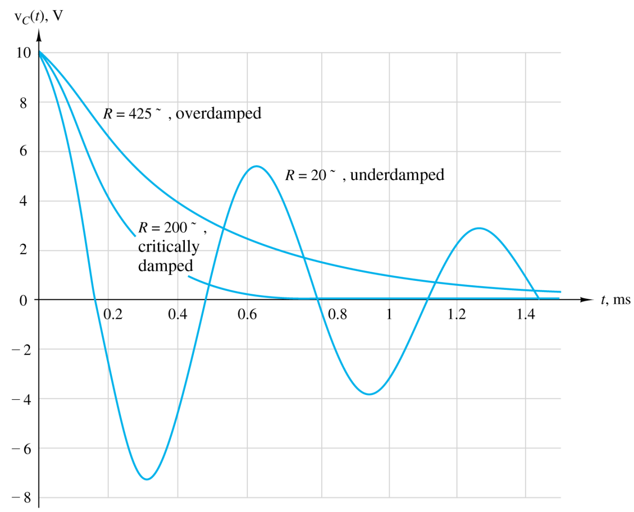
Find i_L for $t > 0$.



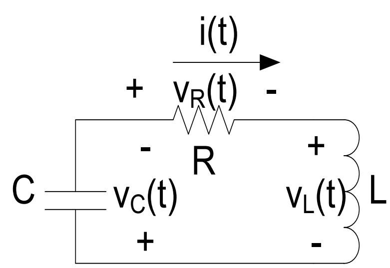


$i_L = e^{-1.2t}(2.027\cos 4.75t + 2.561\sin 4.75t)$ A

Summary of Transient Responses



Source - Free Series RLC Circuit



$$v_R + v_L + v_C = 0$$

$$i(t)R + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt = 0$$

$$L\frac{d^2 i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 0$$

Comparing Series and Parallel RLC Circuits

Parallel RLC	Series RLC
$C\frac{d^2v(t)}{dt^2}+\frac{1}{R}\frac{dv(t)}{dt}+\frac{1}{L}v(t)=0$	$L\frac{d^2i(t)}{dt^2}+R\frac{di(t)}{dt}+\frac{1}{C}i(t)=0$
$v(t)=A_1e^{s_1t}+A_2e^{s_2t}$	$i(t)=A_1e^{s_1t}+A_2e^{s_2t}$
$s_{1,2}=-\frac{1}{2RC}\pm\sqrt{\left(\frac{1}{2RC}\right)^2-\frac{1}{LC}}$	$s_{1,2}=-\frac{R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$
$s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$	$s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$
$\alpha=\frac{1}{2RC}$ $\quad\omega_0=\frac{1}{\sqrt{LC}}$	$\alpha=\frac{R}{2L}$ $\quad\omega_0=\frac{1}{\sqrt{LC}}$

Series RLC Circuit Solution

$$\omega_0={1}/{\sqrt{LC}} \qquad \alpha=\frac{R}{2L} \qquad \begin{aligned} s_1&=-\alpha+\sqrt{\alpha^2-\omega_0^2}\\ s_2&=-\alpha-\sqrt{\alpha^2-\omega_0^2}\end{aligned}$$

If:

$\alpha > \omega_0$ (overdamped): $\quad i(t)=A_1e^{s_1t}+A_2e^{s_2t}$

$\alpha = \omega_0$ (critically damped): $\quad i(t)=e^{-\alpha t}\left(A_1t+A_2\right)$

$\alpha < \omega_0$ (underdamped): $\quad i(t)=e^{-\alpha t}\left(B_1\cos(\omega_d t)+B_2\sin(\omega_d t)\right)$

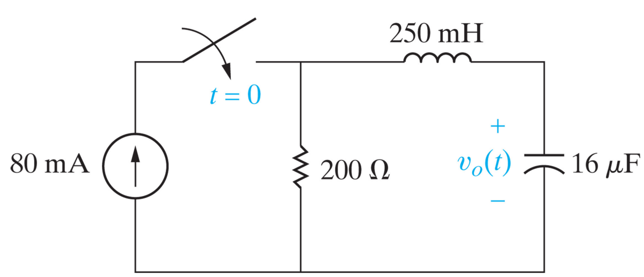
Equations for Analysing the Step Response of Parallel RLC Circuits

Characteristic equation	$s^2 + \frac{1}{RC}s + \frac{1}{LC} = \frac{I}{LC}$
Neper, resonant, and damped frequencies	$\alpha = \frac{1}{2RC} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Roots of the characteristic equation	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
$\alpha^2 > \omega_0^2$: overdamped	$i_L(t) = I_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}, \quad t \geq 0$
	$i_L(0^+) = I_f + A_1' + A_2' = I_0$
	$\frac{di_L(0^+)}{dt} = s_1 A_1' + s_2 A_2' = \frac{V_0}{L}$
$\alpha^2 < \omega_0^2$: underdamped	$i_L(t) = I_f + B_1' e^{-\alpha t} \cos \omega_d t + B_2' e^{-\alpha t} \sin \omega_d t, \quad t \geq 0$
	$i_L(0^+) = I_f + B_1' = I_0$
	$\frac{di_L(0^+)}{dt} = -\alpha B_1' + \omega_d B_2' = \frac{V_0}{L}$
$\alpha^2 = \omega_0^2$: critically damped	$i_L(t) = I_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}, \quad t \geq 0$
	$i_L(0^+) = I_f + D_2' = I_0$
	$\frac{di_L(0^+)}{dt} = D_1' - \alpha D_2' = \frac{V_0}{L}$

(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)

Textbook Problem 8.50 (Nilsson 11th)

The circuit contains no initial energy. Find $v_o(t)$ for $t \geq 0$.



$v_o(t) = 16 - 16e^{-400t}\cos 300t - 21.33e^{-400t}\sin 300t \ V$

Summary: Solving RLC Circuits

1. Identify the series or parallel RLC circuit;
2. Find α and ω_0;
3. Determine whether the circuit is overdamped, critically damped, or underdamped;
4. Assume a solution (natural response + forced response):

$$A_1 e^{s_1 t} + A_2 e^{s_2 t} + V_f \quad \text{Overdamped}$$

$$A_1 t e^{st} + A_2 e^{st} + V_f \quad \text{Critically damped}$$

$$e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t) + V_f \quad \text{Underdamped}$$

5. Find A, B, and V_f using initial and final conditions.

Chapter 8 Summary

- Showed how to determine the natural and the step response of parallel RLC circuits;
- Showed how to determine the natural and the step response of series RLC circuits.