Chapter 4

Engr228
Circuit Analysis
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Chapter 4 Objectives

• Use the node-voltage method to solve a circuit;
• Use the mesh-current method to solve a circuit;
• Determine which technique is best for a particular circuit;
• Be able to use source transformations to simplify a circuit;
• Understand Thévenin and Norton equivalent circuits and be able to derive one;
• Know the condition for maximum power transfer to a load.
JOIN THE RESISTANCE!

Oh MMMMM...

Resistance is futile
Circuit Analysis

• We need an organized method of applying KVL, KCL, and Ohm’s law;
• *Nodal* analysis assigns *voltages* to each node, and then we apply *Kirchhoff's Current Law* to solve for the *node voltages*;
• *Mesh* analysis assigns *currents* to each mesh, and then we apply *Kirchhoff’s Voltage Law* to solve for the *mesh currents*.

Review - Nodes, Paths, Loops, Branches

• These two circuits are equivalent.
• There are three *nodes* and five *branches*:
  – *Node*: a point at which two or more elements have a common connection;
  – *Path*: a sequence of nodes;
  – *Branch*: a single path in a circuit composed of one simple element and the node at each end of that element;
  – *Loop*: a closed path.
Review - Kirchhoff’s Current Law

- Kirchhoff’s Current Law (KCL) states that the algebraic sum of all currents entering a node is zero.

\[ i_A + i_B + (-i_C) + (-i_D) = 0 \]

Review - Kirchhoff’s Voltage Law

- Kirchhoff’s Voltage Law (KVL) states that the algebraic sum of the voltages around any closed path is zero.

\[ -v_1 + v_2 + v_3 = 0 \]
Node Example

- Node = every point along the same wire

3 nodes

Nodes

- How many nodes in the circuits below?
Notes on Writing Nodal Equations

- All terms in the equations are in units of current;
- Everyone has their own style of writing nodal equations
  – The important thing is that you remain consistent.
- Probably the easiest method if you are just getting started is to remember that:
  \[
  \text{current entering a node} = \text{current leaving the node}
  \]
- Current directions can be assigned arbitrarily, unless they are previously specified.

The Nodal Analysis Method

- Assign voltages to every node relative to a reference node.
Choosing the Reference Node

- By convention, the bottom node is often the reference node;
- If a ground connection is shown, then that becomes the reference node;
- Otherwise, choose a node with many connections;
- Assign the reference node a value of 0.00 volts.

Apply KCL to Find Voltages

- Assume reference voltage = 0.00 volts;
- Assign current names and directions;
- Apply KCL to node \( v_1 \) (\( \Sigma \) out = \( \Sigma \) in);
- Apply Ohm’s law to each resistor.

\[
\frac{v_1}{2} + \frac{v_1 - v_2}{5} = 3.1
\]
Apply KCL to Find Voltages

- Apply KCL to node $v_2$ ($\Sigma \text{out} = \Sigma \text{in}$);
- Apply Ohm’s law to each resistor.

We now have two equations for the two unknowns $v_1$ and $v_2$ and we can solve them simultaneously:

$$\frac{v_1 - v_2}{5} = \frac{v_2 - 0}{1} + (-1.4)$$

$\therefore v_1 = 5V$ and $v_2 = 2V$

Example Problem Page 93 - Nilsson 10th

Solve for the node voltages in the circuit below.

Answer: Left node = 9.09 V  
Right node = 10.91 V
**Nodal Analysis: Dependent Source Example**

Determine the power supplied by the dependent source.

Key step: eliminate $i_1$ from the equations using $v_j = 2i_1$.

\[
15 = \frac{v_1 - v_2}{1} + \frac{v_1 - 0}{2}
\]

\[
\frac{v_1 - v_2}{1} + 3i_1 = \frac{v_2 - 0}{3}
\]

\[
i_1 = \frac{v_1 - 0}{2}
\]

*Answer: 4.5 kW being generated*

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**Node Voltage Example**

- How many nodes are in this circuit?
- How many nodal equations must you write to solve for the unknown voltages?

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![Node Voltage Example Diagram](image-url)
Node Voltage Example – Node V1

At node \( V_1 \)
\[
8 + \frac{V_1 - V_3}{4} + 3 + \frac{V_1 - V_2}{3} = 0
\]
\[
96 + 3V_1 - 3V_3 + 36 + 4V_1 - 4V_2 = 0
\]
\[
7V_1 - 4V_2 - 3V_3 = -132
\]

Node Voltage Example – Node V2

At node \( V_2 \)
\[
\frac{V_2 - V_1}{3} - 3 + \frac{V_2 - V_3}{2} + \frac{V_2 - 0}{1} = 0
\]
\[
2V_2 - 2V_1 - 18 + 3V_2 - 3V_3 + 6V_2 = 0
\]
\[
-2V_1 + 11V_2 - 3V_3 = 18
\]
Node Voltage Example – Node V3

At node $V_3$:

\[
\frac{V_3 - V_2}{2} + \frac{V_3 - V_1}{4} - 5\frac{V_3 - 0}{5} = 0
\]

\[
10V_3 - 10V_2 + 5V_3 - 5V_1 - 500 + 4V_3 = 0
\]

\[
-5V_1 - 10V_2 + 19V_3 = 500
\]

Node Voltage Example – Solution

\[
7V_1 - 4V_2 - 3V_3 = -132
\]

\[
-2V_1 + 11V_2 - 3V_3 = 18
\]

\[
-5V_1 - 10V_2 + 19V_3 = 500
\]

Answer:

\[
V_1 = 0.956V
\]

\[
V_2 = 10.576V
\]

\[
V_3 = 32.132V
\]
Voltage Sources and the Supernode

If there is a DC voltage source between two non-reference nodes the current through the voltage source may not be known and an equation cannot be written for it. Therefore, we create a supernode.

The Supernode Analysis Technique

- Apply KCL at Node $v_f$;
- Apply KCL at the supernode;
- Add the equation for the voltage source inside the supernode.

\[
\frac{v_1 - v_2}{4} + \frac{v_1 - v_3}{3} = -3 - 8
\]
\[
\frac{v_1 - v_2}{3} + \frac{v_1 - v_3}{4} = -3 + \frac{v_2}{1} + \frac{v_3}{5} - 25
\]

$v_1 = 1.0714 V$
$v_2 = 10.5 V$
$v_3 = 32.5 V$

$v_3 - v_2 = 22$
Choose the reference point wisely and solve for the currents in the circuit below.

Answer:  
\[ i_a = 0.1 A \]  
\[ i_b = 0.3 A \]  
\[ i_c = 0.2 A \]

Mesh Analysis: Nodal Alternative

- A mesh is a loop that does not contain any other loops within it;
- In mesh analysis, we assign mesh currents and solve using KVL;
- All terms in the equations are in units of voltage;
- Remember – voltage drops in the direction of current flow except for sources that are generating power;
- The circuit below has four meshes:
The Mesh Analysis Method

Mesh currents

Branch currents

Mesh Analysis Example

Apply KVL to mesh 1
( Σ voltage drops = 0 ):

\[-42 + 6i_1 + 3(i_1 - i_2) = 0\]

\[i_1 = 6A\]
\[i_2 = 4A\]

Apply KVL to mesh 2
( Σ voltage drops = 0 ):

\[3(i_2 - i_1) + 4i_2 - 10 = 0\]

\[i_1 = 6A\]
\[i_2 = 4A\]
Textbook Problem 4.32 - Nilsson 10th

Use mesh currents to solve for the currents in the circuit below.

Answer:  
\[ i_a = 0.1A \]
\[ i_b = 0.3A \]
\[ i_c = 0.2A \]

Current Sources and the Supermesh

If a current source is present in the network and shared between two meshes you must use a *supermesh* formed from the two meshes that have the shared current source.
Supermesh Example

Use mesh analysis to find Vx

\[
\begin{align*}
V_x &= 7A \\
I_1 &= I_2 + I_3 \\
I_2 &= I_1 - 2I_3 \\
I_3 &= I_1 - I_2
\end{align*}
\]

Supermesh Example - continued

Loop 2: \( I_1(I_2 - I_1) + 2I_2 + 3(I_2 - I_3) = 0 \)
\(-I_1 + 6I_2 - 3I_3 = 0 \quad Equation \ I \)
Supermesh Example - continued

\[ I_1 = 9A \]
\[ I_2 = 2.5A \]
\[ I_3 = 2A \]
\[ V_x = 3(I_3 - I_2) = -1.5V \]

\[ -7 + I_1(I_1 - I_2) + 3(I_3 - I_2) + I_3 = 0 \]
\[ I_1 - 4I_2 + 4I_3 = 7 \quad \text{Equation II} \]
\[ I_1 - I_3 = 7 \quad \text{Equation III} \]

Node or Mesh: How to Choose?

- Use the one with fewer equations, or
- Use the method you like best, or
- Use both, and check your answers.
Dependent Source Example

Find $V_x$

\[ V_x = 3(17 - 11) = 18V \]

Dependent Source Example - continued

$I_1 = 15A$  \(\text{Equation I}\)

\[ 1(I_2 - I_1) + 2I_2 + 3(I_2 - I_3) = 0 \]

\[ -I_1 + 6I_2 - 3I_3 = 0 \quad \text{Equation II} \]

$I_3 - I_1 = \frac{1}{9}V_x$  \(\text{Equation III}\)

$V_x = 3(I_3 - I_2)$  \(\text{Equation IV}\)

$I_1 = 15A$, $I_2 = 11A$, $I_3 = 17A$

$V_x = 3(17 - 11) = 18V$
Textbook Problem 4.56 Nilsson - 10th

Find the power absorbed by the 20V source in the circuit below.

\[ \text{Power}_{20V} = 480 \text{ mW absorbed} \]

Source Transformation

- The circuits (a) and (b) are equivalent at the terminals;
- If given circuit (a), but circuit (b) is more convenient, switch them;
- This process is called source transformation.
Example: Source Transformation

We can find the current $I$ in the circuit below using a source transformation.

$$I = \frac{45 - 3}{5 + 4.7 + 3} = 3.307 \text{ mA}$$

Example Problem

Find the power of the 7V source.

$$P_{7V} = 17.27W \text{ (generating)}$$
Thévenin Equivalent Circuits

Thévenin’s theorem: a linear network can be replaced by its Thévenin equivalent circuit, as shown below:

Thévenin Equivalent using Source Transformations

- We can repeatedly apply source transformations on network A to find its Thévenin equivalent circuit;
- This method has limitations – due to circuit topology, not all circuits can be source transformed.
Finding the Thévenin Equivalent

- Disconnect the load;
- Find the open circuit voltage $v_{oc}$;
- Find the equivalent resistance $R_{eq}$ of the network with all independent sources turned off.
  - Set voltage sources to zero volts $\rightarrow$ short circuit
  - Set current sources to zero amps $\rightarrow$ open circuit

Then:

$V_{TH} = v_{oc}$ and $R_{TH} = R_{eq}$

Thévenin Example #1
Thévenin Example #2

Find Thévenin’s equivalent circuit and the current passing thru RL given that RL = 1Ω.

\[ V_{TH} = \frac{3}{2+3} \times 10 = 6V \]
**Thévenin Example #2 - continued**

Find $R_{TH}$

$R_{TH} = 10 + 2 \parallel 3 + 2$

$= 10 + \frac{2 \times 3}{2 + 3} + 2$

$= 13.2 \Omega$

**Thévenin Example #2 - continued**

Thévenin’s equivalent circuit

The current thru $RL = 1 \Omega$ is

$$\frac{6}{13.2 + 1} = 0.423 A$$
Example Problem

Find the Thévenin equivalent of the circuit below.

\[ R_{TH} = 8.523 \, k\Omega \]
\[ V_{TH} = 83.5 \, V \]

Norton Equivalent Circuits

Norton’s theorem: a linear network can be replaced by its Norton equivalent circuit, as shown below:
Finding the Norton Equivalent

- Replace the load with a short circuit;
- Find the short circuit current $i_{sc}$;
- Find the equivalent resistance $R_{eq}$ of the network with all independent sources turned off (same as Thévenin)
  - Set voltage sources to zero volts → short circuit;
  - Set current sources to zero amps → open circuit.

Then:
$I_N = i_{sc}$ and $R_N = R_{eq}$

Source Transformation: Norton and Thévenin

The Thévenin and Norton equivalents are source transformations of each other.
Example - Norton and Thévenin

Find the Thévenin and Norton equivalents for the network faced by the 1-kΩ resistor.

This is the circuit we will simplify

Example - Norton and Thévenin - continued

Thévenin

Norton

Source Transformation
Textbook Problem 4.66 Nilsson 10th

Find the Norton Equivalent with respect to terminals a,b.

\[ I_{\text{Norton}} = 7A \]
\[ R_{\text{Norton}} = 16\Omega \]

Thévenin Example: Handling Dependent Sources

The normal technique for finding Thévenin or Norton equivalent circuits can not usually be used if a dependent source is present. In this case, we can find both \( V_{TH} \) and \( I_N \) and solve for \( R_{TH} = V_{TH} / I_N \)
Thévenin Example: Handling Dependent Sources

Another situation that rarely arises, is if both $V_{TH}$ and $I_N$ are zero, or just $I_N$ is zero. In this situation, we can apply a test source to the output of the network and measure the resulting short-circuit ($I_N$) current, or open-circuit voltage ($V_{TH}$). $R_{TH}$ is then calculated as $V_{TH}/I_N$.

Solve: $v_{test} = 0.6$ V, so $R_{TH} = 0.6$ Ω

$$v_{test} = \frac{v_{test}}{2} + \frac{v_{test} - (1.5i)}{3} = 1$$

$$i = -1$$

$0.6$ Ω
Recap: Thévenin and Norton

Thévenin’s equivalent circuit

\[ R_{TH} = R_N \]
\[ V_{TH} = I_N \times R_{TH} \]

Norton’s equivalent circuit

\[ R_{TH} = R_N \]
\[ 6 = 0.45 \times 13.2 \]

Same R value

Maximum Power Transfer

Thévenin’s or Norton’s equivalent circuit delivers a maximum power to the load \( R_L \) for which \( R_{TH} = R_L \)
Maximum Power Theorem Proof

\[ P = I^2 R_L \quad \text{and} \quad I = \frac{V_{TH}}{R_{TH} + R_L} \]

Plug it in

\[ P = \left( \frac{V_{TH}}{R_{TH} + R_L} \right)^2 \cdot R_L = \frac{V_{TH}^2 R_L}{(R_{TH} + R_L)^2} \]

\[ \frac{dP}{dR_L} = \frac{(R_{TH} + R_L)^2 V_{TH}^2 - V_{TH}^2 R_L \cdot 2(R_{TH} + R_L)}{(R_{TH} + R_L)^4} = 0 \]

Maximum Power Theorem Proof - continued

\[ \frac{dP}{dR_L} = \frac{(R_{TH} + R_L)^2 V_{TH}^2 - V_{TH}^2 R_L \cdot 2(R_{TH} + R_L)}{(R_{TH} + R_L)^4} = 0 \]

\[ (R_{TH} + R_L)^2 V_{TH}^2 = V_{TH}^2 R_L \cdot 2(R_{TH} + R_L) \]

\[ (R_{TH} + R_L)^2 = 2R_L \]

\[ R_{TH} = R_L \]

For maximum power transfer
**Maximum Power Example**

Evaluate RL for maximum power transfer and find the power.

![Circuit Diagram]

**Maximum Power Example - continued**

Thévenin’s equivalent circuit

![Circuit Diagram]

RL should be set to 13.2Ω to get maximum power transfer.

Max. power is \[
\frac{V^2}{R} = \frac{(6/2)^2}{13.2} = 0.68W
\]
Chapter 4 Summary

- Illustrated the node-voltage method to solve a circuit;
- Illustrated the mesh-current method to solve a circuit;
- Discussed how to choose which technique is better for a particular circuit;
- Explained source transformations and how to use them to simplify a circuit;
- Illustrated the techniques of constructing Thevenin and Norton equivalent circuits;
- Explained the principle of maximum power transfer to a resistive load and showed how to calculate the value of the load resistor that satisfies this condition.