MULTIPLICATION ALGORITHMS

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RHIND PAPYRUS METHOD

• Method discovered sometime in 1850s
• Document purchased from scavenger by Alexander Henry Rhind in 1858
• Method that doubles and uses addition
EXAMPLE

$22 \times 44 = 968$

1....44
2....88 +
4....176 +
8....352
16....704 +
$704 + 176 + 88 = 968$

PEASANT MULTIPLICATION

- Also known as Egyptian Multiplication
- Only need to know how to add and divide by two
- Can prove helpful due to only dividing by two
EXAMPLE

22 \times 44 = 968
22 \ldots 44
11 \ldots 88 +
5 \ldots 176 +
2 \ldots 352
1 \ldots 704 +
704 + 176 + 88 = 968

KARATSUBA’S ALGORITHM

• Discovered in 1960 by Anatoly Karatsuba
• First algorithm asymptotically faster than the “grade school” method
• Uses the idea of divide and conquer
• Written to spite a professor
## Original vs New

<table>
<thead>
<tr>
<th>Original</th>
<th>Karatsuba’s method</th>
</tr>
</thead>
<tbody>
<tr>
<td>$xy = z_2B^2 + z_1B + z_0$</td>
<td>$z_1 = (x_1 + x_0)(y_1 + y_0) - x_1y_1 - x_0y_0$</td>
</tr>
<tr>
<td>Where</td>
<td></td>
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<tr>
<td>$z_2 = x_1y_1$</td>
<td></td>
</tr>
<tr>
<td>$z_1 = x_1y_0 + x_0y_1$</td>
<td></td>
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<tr>
<td>$z_0 = x_0y_0$</td>
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</tr>
</tbody>
</table>

## Example

$$z_2 = x_1y_1$$

$$z_1 = (x_1 + x_0)(y_1 + y_0) - x_1y_1 - x_0y_0$$

$$z_0 = x_0y_0$$

- Let $x = 112$ and $y = 960$
- $z_2 = 1 \times 9 = 9$
- $z_0 = 12 \times 60 = 720$
- $z_1 = (1 + 12)(9 + 60) - 72 - 720$
- $z_1 = 13 \times 69 - 9 - 720$
- $z_1 = 897 - 9 - 720$
- $z_1 = 168$

$$xy = z_2B^2 + z_1B + z_0$$

$xy = z_2 \times 100^2 + z_1 \times 100 + z_0$

$xy = 90,000 + 16,800 + 720$

$xy = 107,520$
TOOM-COOK

- Karatsuba Algorithm has a problem with recursion
- 1963 – Andrei Toom published his new algorithm
- 1966 – Steven Cook updated the algorithm, made it more efficient
- Works by breaking itself apart into three parts

EXAMPLE: PART 1

- Let A = 123,456,789 and B = 987,654,321
- Split them into three parts
  - A(x) = 123x^2 + 456x + 789
  - B(x) = 987x^2 + 654x + 321
- Pick points for x
  - X = -2; A = 369, B = 2961, AB = 1092609
  - X = -1; A = 456, B = 654, AB = 298224
  - X = 0; A = 789, B = 321, AB = 253269
  - X = 1; A = 1368, B = 1962, AB = 2684016
  - X = 2; A = 2193, B = 5577, AB = 12230361
PART 2

• The product for this multiplication will come in the form of:
  \[ P(x) = p_4x^4 + p_3x^3 + p_2x^2 + p_1x + p_0 \]

• Now plug in all the values of AB into equations relative to their x:
  - \[ 16p_4 - 8p_3 + 4p_2 - 2p_1 + p_0 = 1092609 \]
  - \[ p_4 - p_3 + p_2 - p_1 + p_0 = 298224 \]
  - \[ p_0 = 253269 \]
  - \[ p_4 + p_3 + p_2 + p_1 + p_0 = 2684016 \]
  - \[ 16p_4 + 8p_3 + 4p_2 + 2p_1 + p_0 = 12230361 \]

• Now solve the system of linear equations

PART 3

• From that system of equations we obtain:
  - \[ p_0 = 253269 \]
  - \[ p_1 = 662382 \]
  - \[ p_2 = 1116450 \]
  - \[ p_3 = 530514 \]
  - \[ p_4 = 121401 \]

• Plug these numbers back into: \( P(x) = p_4x^4 + p_3x^3 + p_2x^2 + p_1x + p_0 \)
  - With \( x = 1000 \)
  - \( P(1000) = 121401x^4 + 530514x^3 + 1116450x^2 + 662382x + 253269 \)

• Gives us a final result of: 121,932,631,112,635,269
KARATSUBA VS TOOM-COOK

- Karatsuba runtime: $O(n^{1.585})$
- Toom – Cook runtime: $O(n^{1.465})$
- Pressed for space? Karatsuba
- Pressed for time? Toom – Cook

BIBLIOGRAPHY