Twenty-Sixth Annual
WWU University Days
Mathematics Competition
2013

Winners — Past Five Years		
	1st	**2nd**
2008	WWVA	UCA
2009	UCA	MAA
2010	WWVA	UCA
2011	CCA	MEA
2012	UCA	MEA

team member(s)	head math teacher	academy/h.s.

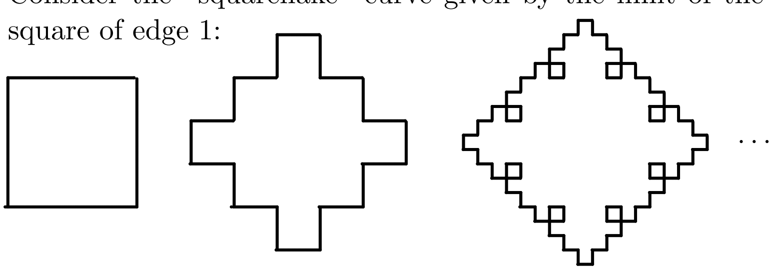
This is a *TEAM* competition. Your answers will be graded based both on **CORRECTNESS** and on the **QUALITY** of your presentation. Show your work *neatly* and indicate your answers *clearly.* Answers *without* justification will receive *no* credit. Please place *no more than* one (1) solution per page. When you are done, arrange your *team* solutions in numerical order (1, 2, 3, ..., 10) before you hand them in. Good luck!!

1. The number $2013 = 3 \cdot 11 \cdot 61$. Subtract 1 from each of these factors and collect all the *prime* factors of these three new numbers. How many distinct products are possible using one or more of this collection of primes?

2. A company sells detergent in three different sized boxes: small (S), medium (M), and large (L). The medium size costs 50% more than the small size and contains 20% less detergent than the large size. The large size contains twice as much detergent as the small size and costs 30% more than the medium size. Rank the three sizes from "best buy" to "worst buy," e.g., LMS.

3. Alice and Bob play a game involving a circle whose circumference is divided by twelve equally spaced points. the points are numbered clockwise, from 1 through 12. Both start on point 12. Alice moves clockwise, and Bob moves counterclockwise. In each turn of the game, Alice moves through 5 points, and Bob moves through 9 points. The game ends when they stop at the same point. How many turns will this take?

4. Consider the "squareflake" curve given by the limit of the following sequence of figures which start with a square of edge 1:



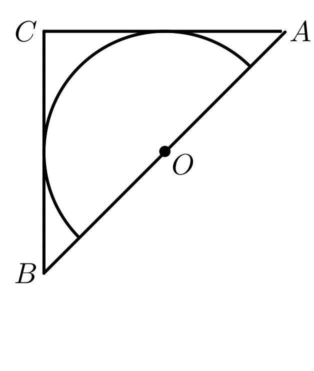
Notice that at each stage there is a "tab" on each edge that sticks out 1/3 and has width 1/3 of the length of each edge of the previous stage. The new tabs *never* overlap each other. Notice also that stage 3 has 8 square holes. Find the area of the region inside the "limiting" curve.

5. Find an equation of the parabola, whose graph is a function y of x, passing through the points $(1, 2)$, $(3, -2)$, and $(2, 1)$.

6. John Gault bought a one-year-old automobile. John wanted to use only pure synthetic motor oil but, rather than change the oil all at once, he thought that he would change it one quart at a time in the following way: Every 1000 miles he would remove the old oil filter with its one quart of oil, put on a new empty filter, and then add a quart of the new synthetic oil. Assuming that the engine and filter hold a total of 5 quarts of oil, how many one-quart changes will he have to make in order for the engine to have at least 3 quarts of the synthetic oil?

7. Sketch the graph of $|x|^{1/3} + |y|^{1/3} = 1$. Justify your graph!

8. An isosceles right triangle ABC encloses a semicircle of area 2π. The circle has its center O on the hypotenuse $\overline{AB}$ and is tangent to sides $\overline{AC}$ and $\overline{BC}$. What is the area of triangle ABC?



9. Solve: $|5 - x| < |x + 4|$.

10. Solve: $\cos 2t + \sin t = 0$.

Solutions: (Remember: No credit for guessing!)

1. We have 2, $10 = 2 \cdot 5$, and $60 = 2^2 \cdot 3 \cdot 5$. Consider first the factors 2, 4, 3, 5. The number of distict products is

$$\binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^4 - 1 = 15.$$

Note that these products also include factors of 8 but neither 16 nor 25. To get the products with a factor 25 and not 16, we simply count all products involving 2, 4, and 3 for another $2^3 = 8$. Now, toss in the two products $16 \cdot 25$ and $16 \cdot 3 \cdot 25$. This completes the products with a factor of 25. There are remaining 4 products involving 16 and any combination of 3 and 5 for a total of $15 + 8 + 2 + 4 = 29$ products.

2. Neither the units of size nor the costs are important to this problem. So for convenience, we suppose that the small box costs 1$ and weighs 10 ounces. Then we compare their relative value using cost per unit of weight:

$$S : \frac{\$1}{10} = \$0.10 \text{ per oz.} \quad M : \frac{\$1.50}{16} = \$0.09375 \text{ per oz.} \quad L : \frac{1.3 \times \$1.50}{20} = \$0.0975 \text{ per oz.}$$

The correct answer is MLS.

3. The game will end after 6 turns. Observe the points where each player is after each turn:

$$\begin{aligned} Alice :& 12, \quad 5, \quad 10, \quad 3, \quad 8, \quad 1, \quad 6 \\ Bob :& 12, \quad 3, \quad 6, \quad 9, \quad 12, \quad 3, \quad 6 \end{aligned}$$

4. The area is

$$\begin{aligned} A &= 1 + 4 \cdot \left(\frac{1}{3}\right)^2 + 5 \cdot 4 \cdot \left(\left(\frac{1}{3}\right)^2\right)^2 + 5^2 \cdot 4 \cdot \left(\left(\left(\frac{1}{3}\right)^2\right)^2\right)^2 + \cdots \\ &= 1 + 4 \cdot \frac{1}{9} + 5 \cdot 4 \cdot \left(\frac{1}{9}\right)^2 + 5^2 \cdot 4 \cdot \left(\frac{1}{9}\right)^3 + \cdots \\ &= 1 + \frac{4}{9} + \frac{4}{9} \cdot \frac{5}{9} + \frac{4}{9} \cdot \left(\frac{5}{9}\right)^2 + \cdots \\ &= 1 + \frac{4/9}{1 - 5/9} \\ &= 1 + 1 = 2. \end{aligned}$$

5. So, $y = ax^2 + bx + c$. The points are not colinear. Fill in for x and y and get the system:

$$\begin{aligned} 2 &= a + b + c \\ -2 &= 9a + 3b + c \\ 1 &= 4a + 2b + c \end{aligned}$$

Solve and get: $a = -1$, $b = 2$, and $c = 1$.

6.

Oil change	1	2	3	4	5
Syn out	0	$\frac{1}{5}$	$\frac{1}{5} \cdot \frac{9}{5}$	$\frac{1}{5} \cdot \frac{61}{25}$	$\frac{1}{5} \cdot \frac{369}{125}$
Syn in	1	1	1	1	1
Tot Syn	1	$\frac{9}{5}$	$\frac{4 \cdot 9}{25} + \frac{25}{25}$	$\frac{4 \cdot 61}{125} + \frac{125}{125}$	$\frac{4 \cdot 369}{625} + \frac{625}{625}$

Now note that $\frac{369}{125} < 3$ but $\frac{4 \cdot 369 + 625}{625} = \frac{2101}{625} > \frac{1875}{625} = 3$. Whew!

7. First note that the graph is symmetric w.r.t. both axes. Note also that the points $(0, 1)$ and $(1, 0)$ lie on the graph as do the points $(1/8, 1/8)$, $(1/27, 8/27)$, and $(8/27, 1/27)$. Connect with a smooth curve in the first quadrant. Reflect graph appropriately.

8. 8. (Nice, huh!) OK, OK: The area of the semicircle is 2π so the area of the whole circle is $4\pi = \pi r^2$. Thus the radius of the circle is 2. Hence, the circle has diameter $d = 4$ which means the triangle has area $A = \frac{1}{2} \cdot 4 \cdot 4 = 8$.

9. Set up a signed graph for the quantities without absolute value. Note that they are equal at $x = 1/2$. Note also that the original inequality can never hold for negative x. The solution interval then is $(1/2, \infty)$.

10. We solve $1 - 2\sin^2 t + \sin t = 0$ or, $2\sin^2 t - \sin t - 1 = 0$. Thus $(2\sin t + 1)(\sin t - 1) = 0$ or $\sin t = -1/2$ and $\sin t = 1$. Hence, $t = 7\pi/6 + 2k\pi$ and $t = 11\pi/6 + 2k\pi$ where k is any integer. For the other piece, $\sin t = 1$ so that $t = \pi/2 + 2k\pi$ where, again, k is any integer.