The Silver Anniversary!!

Twenty-Fifth Annual WWU University Days

MATHEMATICS COMPETITION

2012

Winners — Past Five Years	1st	2nd
2007	CAA	UCA
2008	WWVA	UCA
2009	UCA	MAA
2010	WWVA	UCA
2011	CCA	MEA

team member(s)	head math teacher	academy/h.s.

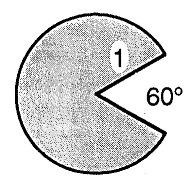
This is a *TEAM* competition. Your answers will be graded based both on **CORRECTNESS** and on the **QUALITY** of your presentation. Show your work *neatly* and indicate your answers *clearly*. Answers *without* justification will receive *no* credit. Please place *no more than* one (1) solution per page. When you are done, arrange your *team* solutions in numerical order (1, 2, 3, . . . , 10) before you hand them in. Good luck!!

1. Let S be the set of the 2012 smallest positive multiples of 4, and let T be the set of the 2012 smallest positive multiples of 6. How many elements are common to the sets S and T?

   (a) 503 (b) 2012 (c) 606 (d) 670 (e) 566

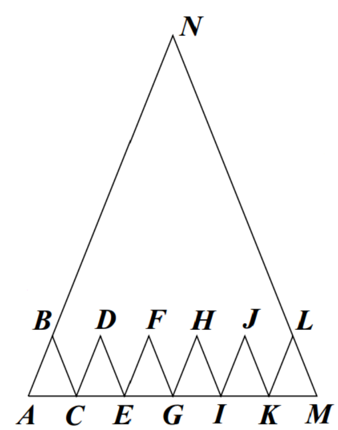
2. An equiangular octagon has four sides of length 1 and four sides of length $\frac{\sqrt{2}}{2}$ arranged so that no two consecutive sides have the same length. What is the area of the octagon?

3. A solid large cube of edge length n ($n \geq 2$) is made up of cubes of edge length 1 (unit cubes). The surfaces of the large cube are then painted red. Find an expression in n which gives the number of unit cubes which have *no* red paint on them.



4. In an arcade game, the "monster" is the shaded sector of a circle of radius 1 cm, as shown in the figure to the right. The missing piece (the mouth) has central angle 60 deg. What is the perimeter of the monster in cm?

5. Given that $\log_{10} 2 = 0.3010299957\ldots$ and $\log_{10} 3 = 0.4771212547\ldots$, what is the number of digits in the decimal expansion of 12^{10}?



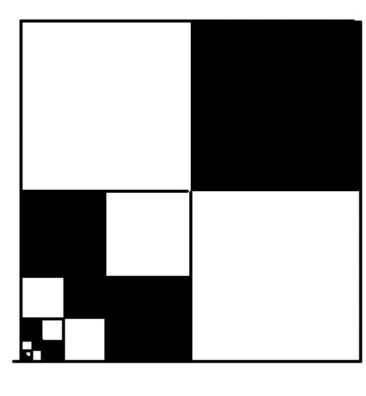
6. A parabola $y = ax^2 + bx + c$ has vertex $(4, 2)$. If $(2, 0)$ is on the parabola, find the product abc.

7. Six distinct integers are picked at random from $\{1, 2, 3, ..., 10\}$. What is the probability that, among those selected, the second smallest is 3?

8. Given that triangles $\triangle ABC$, $\triangle CDE$, $\triangle EFG$, $\triangle GHI$, $\triangle IJK$, and $\triangle KLM$ in the figure to the right are all congruent, and that they are all similar to $\triangle ANM$, what is the ratio of the area of $\triangle ANM$ to $\triangle ABC$?

9. Each integer 1 through 9 is written on a separate slip of paper and all nine slips are put into a hat. Jack picks one of these slips at random and puts it back. Then Jill picks a slip at random. Which digit is most likely to be the units digit of the **sum** of Jack's integer and Jill's integer?

   (a) 0 (b) 1 (c) 8 (d) 9 (e) Each digit is equally likely.



10. Assuming that the square in the given diagram at right-hand side is being divided at its midpoints at each stage, that the pattern continues indefinitely, and that the original edge-length is 1, find the total area of the shaded portion. Note that two quarter-squares are added to the shaded region every-other time.

Solutions:

1. (d) 670 elements (see sheet for calculations)
2. $\frac{7}{2}$ square units (see sheet for geometry and calculations)
3. $n^3 - (6n^2 - 12n + 8)$.
4. $2\pi r - (2\pi r)/6 + 2 = \frac{5\pi}{3} + 2$.
5. The number of digits in the decimal expansion is the ceiling of $\log 12^{10}$. This is:

   $$\log 12^{10} = 10 \log 12 = 10 \log(4{\cdot}3) = 10(\log 4 + \log 3) = 10(2 \log 2 + \log 3) = 10(2(0.301029957) + 0.4771212547) \approx 10(0.602$$

   Therefore, there are 11 digits in the decimal expansion of 12^{10}.
6. By symmetry, $(6, 0)$ is also on the parabola. This gives us three equations: $\begin{array}{rcrcrcl} 16a & + & 4b & + & c & = & 2 \\ 4a & + & 2b & + & c & = & 0 \\ 36a & + & 6b & + & c & = & 0 \end{array}$. The solution is $a = -1/2$, $b = 4$, and $c = -6$ so $abc = 12$.
7. 1/3
8. Note that there are $6 + 5 + 4 + 3 + 2 + 1$ small triangles pointing up and $5 + 4 + 3 + 2 + 1$ small triangles pointing down for a total of $21 + 15 = 36$. Thus the ratio is 36:1. Alt: Since the smaller triangles are all congruent, the measures of AC, CE, EG, GI, IK, and KM are all equal. Call this length y. Then $6y = AM$, and the scaling factor between the smaller triangles and the larger similar triangles is 1:6. Let x be the length of the altitude of $\triangle ABC$ which bisects $\overline{AC}$. Then the length of the altitude of $\triangle ANM$ bisecting $\overline{AM}$ is $6x$. Therefore, the ratio of the area of $\triangle ANM$ to the area of $\triangle ABC$ is:

   $$\frac{0.5(6y)(6x)}{0.5(yx)} = \frac{36}{1}$$

   or $36 : 1$.
9. (a) Regardless of the digit Jack chooses, the sum 10 is always possible. No other sum is *always* possible.
10. $A = \left(\frac{1}{2}\right)^2 + 2 \cdot \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + 2 \cdot \left(\frac{1}{2^4}\right)^2 + \left(\frac{1}{2^5}\right)^2 + 2 \cdot \left(\frac{1}{2^6}\right)^2$ ...+ which yields two geometric series:

    $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2^3}\right)^2 + \left(\frac{1}{2^5}\right)^2 + \cdots = \frac{1/4}{1 - 1/16}$ and

    $2 \cdot \left[\left(\frac{1}{2^2}\right)^2 + \left(\frac{1}{2^4}\right)^2 + \left(\frac{1}{2^6}\right)^2 + \cdots = \frac{1/16}{1 - 1/64}\right.$ for a total of $\frac{4}{15} + \frac{16}{63} = \frac{80 + 84}{63 \cdot 5} = \frac{164}{315}$.