Twenty-Fourth Annual
WWU University Days
Mathematics Competition
2011

Winners — Past Five Years	1st	2nd
2006	WWVA	PAA
2007	CAA	UCA
2008	WWVA	UCA
2009	UCA	MAA
2010	WWVA	UCA

team member(s)	head math teacher	academy/h.s.

This is a *TEAM* competition. Your answers will be graded based both on **CORRECTNESS** and on the **QUALITY** of your presentation. Show your work *neatly* and indicate your answers *clearly*. Answers *without* justification will receive *no* credit. Please place *no more than* one (1) solution per page. When you are done, arrange your team solutions in numerical order (1, 2, 3, . . . , 10) before you hand them in. Good luck!!

1. For each pair of real numbers, a and b, where $a \neq b$, define $a * b = \dfrac{a+b}{a-b}$. What is the value of $(1 * 2) * 3$?

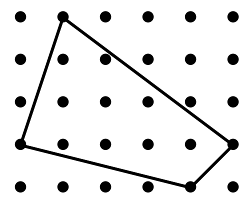
2. While eating out, Joe and Bob each tipped their server \$3. Joe tipped 15% of his bill while generous Bob tipped 20% of his bill. What was the difference, in dollars, between their two bills?

3. Simplify the infinite product $(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\cdots$ given that $|x| < 1$. (Would $1-x$ be useful?)

4. A father in his will left all his money to his children as indicated in the box on the right. When this was done, each child had the same amount of money. How many children were there?

   \$1000 to the first born plus $\frac{1}{10}$th of what remained
\$2000 to the second born plus $\frac{1}{10}$th of what remained
\$3000 to the third born plus $\frac{1}{10}$th of what remained
and so on. . .

5. Pegs are put in a board 1 unit apart both horizontally and vertically. A rubber band is stretched over 4 pegs as shown in the figure at the right, forming a quadrilateral. Find the area of the quadrilateral.



6. How many ordered triples (a, b, c) of non-zero real numbers have the property that each number is the product of the other two?

7. In a mathematics competition, the sum of the scores of Bill and Dick equaled the sum of the scores of Ann and Carol. If the scores of Bill and Carol had been interchanged, the the sum of the scores of Ann and Bill would have exceeded the sum of the scores of the other two. Also, Dick's score exceeded the sum of the scores of Bill and Carol. Determine the order in which the four contestants finished, from highest to lowest. Assume the scores were nonnegative.

8. An isosceles trapezoid is circumscribed about a circle. The lower base of the trapezoid has length 16, and one of the lower base angles is $\arcsin(0.8)$. Find the area of the trapezoid.

9. Solve: $|2x+3| = |x+4| + 1$.

10. How are the numbers 58, 112, and, 2011 related? What are the "bases" of your conclusion?

Solutions:

1. 0: Note that $1 * 2 = \dfrac{1+2}{1-2} = -3$ so $(1*2)*3 = -3*3 = \dfrac{-3+3}{-3-3} = 0$.

2. \$5: Since \$3 is 0.15 of Joe's bill, we know that his bill was \$300/15 or \$20. Similarly, Bob's bill had to be \$15.

3. $\dfrac{1}{1-x}$: Let $P = (1+x)(1+x^2)(1+x^4)(1+x^8)\cdots$. Then

$$\begin{aligned}(1-x)P &= (1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots \\ &= (1-x^2)(1+x^2)(1+x^4)(1+x^8)\cdots \\ &= (1-x^4)(1+x^4)(1+x^8)\cdots \\ &\vdots \\ &= 1,\end{aligned}$$

since $|x| < 1$. Therefore, solving $(1-x)P = 1$ for P yields $P = \dfrac{1}{1-x}$.

4. 9: Let x be the total amount of money bequeathed by the father. Then, the first child got:

$$y = 1000 + \frac{x - 1000}{10}$$

and the second child got

$$y = 2000 + \frac{x - 2000 - y}{10}.$$

Solving this system of equations in two unknowns yields $x = \$81,000$ and $y = \$9,000$. Thus, there were $81000/9000 = 9$ children.

5. 10: Note that each region outside the quadrilateral is a right triangle. The sum of the areas of the four right triangles is: $\frac{1}{2}(1\cdot 3 + 4\cdot 3 + 1\cdot 1 + 1\cdot 4) = 10$. The dotbox has area 20 so $20 - 10 = 10$.

6. 4: The simultaneous equations $a = bc, b = ca, c = ab$ imply $abc = (bc)(ca)(ab) = (abc)^2$, so either $abc = 0$ (ruled out) or $abc = 1$. The same simultaneous equations above also imply $abc = a^2 = b^2 = c^2$, so $|a| = |b| = |c| = 1$. It cannot be that all three unknowns are -1, nor can exactly one be -1, for in either case $a = bc$ is not satisfied. However, the remaining four cases, $(a, b, c) = (1, 1, 1), (-1, -1, 1), (-1, 1, -1), (1, -1, -1)$ are all solutions. Thus there are 4 solutions in all.

7. A, D, B, C: Let A, B, C, and D represent the scores of Ann, Bill, Carol, and Dick, respectively. Then:

$$\begin{aligned} 1)\ & A + C = B + D \\ 2)\ & A + B > C + D \\ 3)\ & D > B + C \end{aligned}$$

Adding 1) and 2) shows $2A > 2D$ or $A > D$. Subtracting 1) from 2) gives $B - C > C - B$ or $B > C$. By 3), $D > B$, so $A > D > B > C$.

8. 80: By viewing the sides of the trapezoid as tangents to the circle, we find that the sums of the lengths of opposite sides are equal. Defining x as the slant side length, and y as the upper base length, we have $2y + 1.2x = 2x$ and $y + 1.2x = 16$. Then $y = 4$ and $x = 10$, and the area is $\frac{1}{2}(4+16)(8) = 80$. There other ways to find the area without having to remember the one fact about circles inscribed in trapezoids.

9. $x = 2,\ -8/3$.

10. 2011 is the base 3 representation of 58 while 112 is 58 in base 7. I.e., $58 = 2011_3 = 112_7$.